Contextual, Optimal, and Universal Realization of the Quantum Cloning Machine and of the NOT Gate

Francesco De Martini, Daniele Pelliccia, and Fabio Sciarrino
Dipartimento di Fisica and Istituto Nazionale per la Fisica della Materia, Università di Roma "La Sapienza," Roma, 00185–Italy
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A simultaneous realization of the universal optimal quantum cloning machine and of the universal-NOT gate by a quantum injected optical parametric amplification, is reported. The two processes, forbidden in their exact form for fundamental quantum limitations, are found universal and optimal, and the measured fidelity $F < 1$ is found close to the limit values evaluated by quantum theory. This work may enlighten the yet little explored interconnections of fundamental axiomatic properties within the deep structure of quantum mechanics.

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A most interesting aspect of quantum information consists of its insightful enlightening of the deep structure of quantum mechanics (QM), the cornerstone of our yet uncertain understanding of the Universe. Because of its key role in various branches of science and of the immediate confrontation with its well alive "classical" counterpart, quantum information (QI) has indeed become the training field for intriguing questions, apparent contradictions and paradoxes, "forbidden" processes. It is then not surprising that after a century long quantum endeavor the inner structure of the theory has been drawn recently to a profound investigation. For instance, several unexpected quantum "bounds" were discovered mostly in quantum measurement theory. A renowned example is the "no-cloning" theorem implying the impossibility of using the Einstein-Podolsky-Rosen (EPR) correlations for the superluminal communication of significant messages [1–3]. Another important, recently discovered forbidden QI process consists of the impossibility of realizing in a two-dimensional Hilbert space the Universal NOT-gate, i.e., a spin-flip map by which any input qubit $|\Psi\rangle$ is transformed into the corresponding orthogonal qubit $|\Psi^{\perp}\rangle$ [4]. In the present Letter these two forbidden processes are realized for the first time "contextually" and "optimally," i.e., with the maximum approximation allowed by QM, by two separate entangled parts of the same apparatus: a quantum injected optical parametric amplifier (QIOPA) [5]. In analogy with the original definition by Kochen and Specker (KS) here "contextuality" implies the interdependence of local measurements on distant systems connected by a bipartite entanglement [6,7]. An unexpected and quite intriguing aspect of this condition is the fact that according to axiomatic quantum theory the "impossibility" of two forbidden processes lies on the violation of two distinct and independent necessary QM requirements, namely, the linearity and the complete positivity (CP) of any QM map [6]. There is evidence that this interesting correlation reflects a very general property of the optimal quantum transposition map in any entangled system.

The two processes were realized simultaneously in a 2×2 dimensional Hilbert space $H_1 \otimes H_2$ of photon polarization ($\vec{\pi}$) and the linearized 1-to-2 cloning process, i.e., leading from $N = 1$ input qubit to $M = 2$ clones, was investigated. Consider first the case of an input $\vec{\pi}$-encoded qubit $|\Psi\rangle_{in}$ associated with a single photon with wavelength (wl) λ, injected on the input mode k_1 of the QIOPA, the other input mode k_2 being in the vacuum state [4]. As for previous works, the photon was injected into a nonlinear (NL) BBO (β-barium-borate) 1.5 mm thick crystal slab, cut for type II phase matching and excited by a sequence of UV mode-locked laser pulses having duration $\tau \approx 140f$ sec and wl λ_p: Fig. 1. The relevant modes of the NL 3-wave interaction driven by the UV pulses associated with mode k_p were the two spatial modes with wave-vector (wv) k_i, $i = 1, 2$, each supporting the two horizontal (H) and vertical (V) linear $\vec{\pi}$s of the interacting photons. The QIOPA was λ degenerate, i.e., the interacting photons had the same wl's

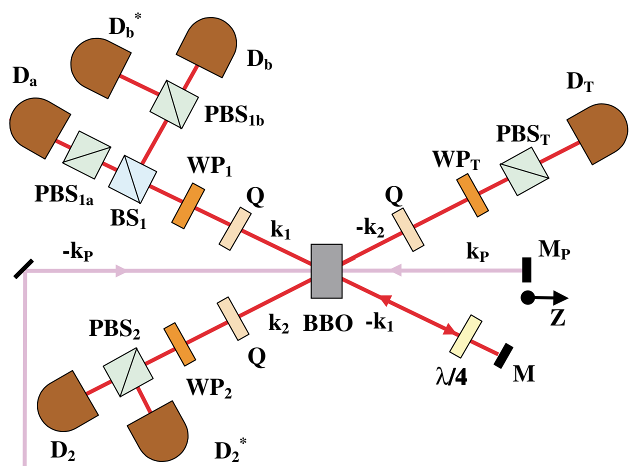

FIG. 1 (color online). Schematic diagram of the universal optimal cloning machine (UOQCM) realized on the cloning (C) channel (mode k_1) of a self-injected OPA and of the Universal NOT (U-NOT) gate realized on the anticloning (AC) channel, k_2.

$\lambda = \frac{1}{2}\lambda_p = 795$ nm. The NL crystal orientation was set as to realize the insensitivity of the amplification quantum efficiency (QE) to any input state $|\Psi\rangle_{\rm in}$, i.e., the universality (U) of the "cloning machine" and of the U-NOT gate under investigation. It is well known that this key property is assured by the squeezing Hamiltonian [4,8]:

$$\hat{H}_{\rm int} = i\chi\hbar(\hat{a}^\dagger_\Psi \hat{b}^\dagger_{\Psi\perp} - \hat{a}^\dagger_{\Psi\perp}\hat{b}^\dagger_\Psi) + {\rm H.c.} \qquad (1)$$

The field operators sets $\{\hat{a}^\dagger_\Psi, \hat{a}_\Psi\}$, $\{\hat{a}^\dagger_{\Psi\perp}, \hat{a}_{\Psi\perp}\}$, $\{\hat{b}^\dagger_\Psi, \hat{b}_\Psi\}$, $\{\hat{b}^\dagger_{\Psi\perp}, \hat{b}_{\Psi\perp}\}$ refer to two mutually orthogonal $\vec{\pi}$ states, $|\Psi\rangle$ and $|\Psi^\perp\rangle$, realized on the two interacting spatial modes k_i. The $\hat{a}$ and $\hat{b}$ operators refer to modes k_1 and k_2, respectively [4]. The $SU(2)$ invariance of $\hat{H}_{\rm int}$ implied by the U condition, i.e., the independence of the OPA "gain" $g \equiv \chi t$ to any unknown $\vec{\pi}$ state of the injected qubit, t being the interaction time, allows the use of the subscripts Ψ and $\Psi^\perp$ in Eq. (1) [8].

The QIOPA apparatus adopted in the present Letter was arranged in the self-injected configuration shown in Fig. 1. The UV pump beam, back reflected by a spherical mirror $\mathbf{M_p}$ with 100% reflectivity and μ-adjustable position $\mathbf{Z}$, excited the NL crystal in both directions $-k_p$ and k_p, i.e., correspondingly oriented towards the right-hand side (rhs) and the left-hand (lhs) side of Fig. 1. A spontaneous parametric down conversion (SPDC) process excited by the $-k_p$ UV mode created singlet states of photon polarization (π). The photon of each SPDC pair emitted over $-k_1$ was back reflected by a spherical mirror $\mathbf{M}$ into the NL crystal and provided the $N = 1$ quantum injection into the OPA excited by the UV beam associated with the back-reflected mode k_p. Because of the low pump intensity, the probability of the unwanted $N = 2$ injection has been evaluated to be 10^{-2} smaller than the one for $N = 1$. The twin SPDC photon emitted over mode $-k_2$, selected by the devices (wave plate + polarizing beam splitter: $WP_T + PBS_T$) and detected by D_T, provided the "trigger" of the overall conditional experiment. Because of the EPR nonlocality of the emitted singlet, the π selection made on $-k_2$ implied deterministically the selection of the input state $|\Psi\rangle_{\rm in}$ on the injection mode k_1. By adopting $\lambda/2$ or $\lambda/4$ wave plates (WP) with different orientations of the optical axis, the following $|\Psi\rangle_{\rm in}$ states were injected: $|H\rangle$, $2^{-1/2}(|H\rangle + |V\rangle)$, and $2^{-1/2}(|H\rangle + i|V\rangle)$. The three fixed quartz plates (Q) inserted on the modes k_1, k_2, and $-k_2$ provided the compensation for the unwanted walk-off effects due to the birefringence of the NL crystal. An additional walk-off compensation into the BBO crystal was provided by the $\lambda/4$ WP exchanging on mode $-k_1$ the $|H\rangle$ and $|V\rangle$ $\vec{\pi}$ components of the injected photon. All adopted photodetectors (D) were equal SPCM-AQR14 Si-avalanche single photon units with $QE \cong 0.55$. One interference filter with bandwidth $\Delta\lambda = 6$ nm was placed in front of each D.

Let us consider the injected photon in the mode k_1 to have any polarization $\vec{\pi} = \Psi$, corresponding to the unknown input qubit $|\Psi\rangle_{\rm in}$. We express this $\vec{\pi}$ state as $\hat{a}^\dagger_\Psi|0,0\rangle_{k_1} = |1,0\rangle_{k_1}$ where $|m,n\rangle_{k_1}$ represents a product state with m photons of the mode k_1 having the polarization Ψ, and n photons having the polarization $\Psi^\perp$. Assume the input mode k_2 to be in the vacuum state. The initial $\vec{\pi}$ state of modes k_i reads $|\Psi\rangle_{\rm in} = |1,0\rangle_{k_1} \otimes |0,0\rangle_{k_2}$ and evolves according to the unitary operator $\hat{\mathbf{U}} \equiv \exp(-i\hat{H}_{\rm int}t)$:

$$\hat{\mathbf{U}}|\Psi\rangle_{\rm in} \simeq |1,0\rangle_{k_1} \otimes |0,0\rangle_{k_2} + g(\sqrt{2}|2,0\rangle_{k_1} \otimes |0,1\rangle_{k_2} - |1,1\rangle_{k_1} \otimes |1,0\rangle_{k_2}). \qquad (2)$$

This represents the first-order approximation for the pure output state vector $|\Psi\rangle_{\rm out}$ for $t > 0$ [4]. Be $\varrho \equiv (|\Psi\rangle\langle\Psi|)_{\rm out}$ the corresponding density operator. It has been shown that the amplified term $\propto g$ in $|\Psi\rangle_{\rm out} = \hat{\mathbf{U}}|\Psi\rangle_{\rm in}$ is equal to the output state $|\Psi_N\rangle_M$ of a general universal optimal quantum cloning machine (UOQCM), where N and $M > N$ are, respectively, the number of input qubits to be cloned and the number of clones [8,9]. The above linearization procedure, i.e., the restriction to the simplest case $N = 1$, $M = 2$, is justified here by the small experimental value of the gain: $g \approx 0.1$. The zero order term in Eq. (2) corresponds to the absence of NL interaction while the second term describes the first-order QIOPA amplification process. In this context, the state $|2,0\rangle_{k_1}$ expressing two photons of the mode k_1 in the $\vec{\pi}$ state Ψ corresponds to the state $|\Psi\Psi\rangle$ expressed by the general theory and implies the $M = 2$ cloning of the input $N = 1$ qubit [1,9,10]. Contextually with the realization of cloning on mode k_1, the vector $|0,1\rangle_{k_2}$ in Eq. (2) expresses the single photon state on mode k_2 with polarization $\Psi^\perp$, i.e., the flipped version of the input qubit. Then the QIOPA acts simultaneously on the output mode k_1 as an approximate UOQCM and on the output mode k_2 as an approximate Universal NOT-gate, i.e., two processes which are forbidden in their exact form, as said [4].

To see that the stimulated emission is indeed responsible for the creation of the flipped qubit, let us compare the state of Eq. (2) with the output of the OPA when the vacuum state is injected into the NL crystal on both input modes k_i. In this case is $|\Psi\rangle^0_{\rm in} = |0,0\rangle_{k_1} \otimes |0,0\rangle_{k_2}$ and we obtain to the first order of approximation: $\hat{\mathbf{U}}|\Psi\rangle^0_{\rm in} \simeq |0,0\rangle_{k_1} \otimes |0,0\rangle_{k_2} + g(|1,0\rangle_{k_1} \otimes |0,1\rangle_{k_2} - |0,1\rangle_{k_1} \otimes |1,0\rangle_{k_2})$. We see that the flipped qubit expressed by the state $|0,1\rangle_{k_2}$ in the right-hand sides of the last equation and of Eq. (2) appears with different amplitudes corresponding to the ratio of probabilities to be equal to $R^* = 2{:}1$. The quantity R^* may be referred to as "signal-to-noise" ratio: S/N. Note also in these equations that, by calling by R the ratio of the probabilities of detecting 2 and 1 photons on mode k_1, we obtain: $R = R^*$. In other words, and highly remarkably the same value of S/N affects both cloning and U-NOT processes realized contextually on the two different output modes: $k_1$and k_2. The ratios R and R^* are indeed the quantities measured in the present experiments for both UOQCM and U NOT-gate processes, respectively. These ratios are used to determine the corresponding values of the fidelity, defined as follows [4,9,10]: Cloning fidelity: $F = \mathrm{Tr}(\rho_1 \hat{n}_{1\pi})/\mathrm{Tr}(\rho_1 \hat{n}_1) = (2R+1)/(2R+2)$. U-NOT fidelity: $F^* = \mathrm{Tr}(\rho_2 \hat{n}_{2\perp})/\mathrm{Tr}(\rho_2 \hat{n}_2) = R^*/(R^*+1)$. There: $\hat{n}_i = \hat{n}_{i\Psi} + \hat{n}_{i\perp}$, being $\hat{n}_{1\Psi} \equiv \hat{a}^\dagger_\Psi \hat{a}_\Psi$, $\hat{n}_{1\perp} \equiv \hat{a}^\dagger_{\Psi\perp} \hat{a}_{\Psi\perp}$, $\hat{n}_{2\Psi} \equiv \hat{b}^\dagger_\Psi \hat{b}_\Psi$, $\hat{n}_{2\perp} \equiv \hat{b}^\dagger_{\pi\perp} \hat{b}_{\pi\perp}$, and the reduced density operators of the output state on the modes k_i are $\varrho_1 = \mathrm{Tr}_2\rho$ and $\varrho_2 = \mathrm{Tr}_1\rho$. The values $F = 5/6$ and $F^* = 2/3$ are the "optimal" values corresponding to the limit values $R = R^* = 2$ allowed by QM, respectively, for cloning $N = 1$ into $M = 2$ qubits and for realizing a U-NOT gate by single qubit flipping.

The goal of the cloning experiment was to measure, under injection of the state $|\Psi\rangle_{\mathrm{in}}$, the S/N related to the OPA amplification leading to the state $|2,0\rangle_{k_1}$ on the output mode k_1, here referred to as the "cloning mode" (C). At the same time on the C mode no amplification should affect the output state $|1,1\rangle_{k_1}$ corresponding to $\vec{\pi}^\perp$ orthogonal to $\vec{\pi}$. In order to perform this task, the PBS_2 was removed on the mode k_2 and the photons on the same mode detected by a single detector: D_2. The output $M = 2$ photons associated with the C mode were separated by means of a 50:50 conventional beam splitter (BS_1) and their $\vec{\pi}$ states analyzed by the combinations of WP_1 and of PBS_{1a} and PBS_{1b}. For each injected $\vec{\pi}$ state, $|\Psi\rangle_{\mathrm{in}}$ WP_1 was set in order to detect $|\Psi\rangle$ by D_a and D_b and to detect $|\Psi^\perp\rangle$, orthogonal to $|\Psi\rangle$ by D_b^*. Hence any coincidence event detected by D_a and D_b corresponded to the realization of the state $|\Psi\Psi\rangle_1$ over the C mode, while a coincidence detected by D_a and D_b^* corresponded to the state $|\Psi\Psi^\perp\rangle_1$.

The measurement of R could be carried out by 4-coincidence measurements involving simultaneously the detector sets: $[D_2, D_T, D_a, D_b]$ and $[D_2, D_T, D_a, D_b^*]$: $R = \frac{1}{2}C_1/C_2$ being C_1 and C_2 the results obtained by the sets, respectively. A better alternative method, actually adopted in the experiment is described in [11].

The experimental data reported on the lhs column of Fig. 2 correspond to the 4-coincidence measurement by $[D_2, D_T, D_a, D_b]$ that is, to the emission over the C mode of the "cloned" state $|\Psi\Psi\rangle_1$ under injection of the state $|\Psi\rangle_{\mathrm{in}}$. The resonance peaks found by this measurement identified the position **Z** of the UV mirror $\mathbf{M_p}$ corresponding to the maximum overlapping of the pump and of the injected pulses, i.e., to the actual realization of the QIOPA operation. According with the analysis above the "noise" plots, implying the realization of the state $|\Psi\Psi^\perp\rangle_1$ measured by the 4-coincidence set $[D_2, D_T, D_a, D_b^*]$, do not show any OPA amplification effect. Each pair of adjacent plots belonging to the two columns in Fig. 2 corresponded to the same injected $\vec{\pi}$ state $|\Psi\rangle_{\mathrm{in}}$. Precisely, starting from the upper pair down, linear-$\vec{\pi}$ horizontal, linear-$\vec{\pi}$ at 45°, left circular-$\vec{\pi}$. The corresponding experimental values of the cloning fidelity were found: $F_H = 0.812 \pm 0.007$; $F_{H+V} = 0.812 \pm 0.006$; $F_{\mathrm{left}} = 0.800 \pm 0.007$, to be compared with the optimal values $F_{\mathrm{th}} = 5/6 \approx 0.833$ corresponding to the limit value: $R = 2$. This result is consistent with the one obtained by A. Lamas-Linares *et al.* by adoption of a semiclassical, coherent injected field [10].

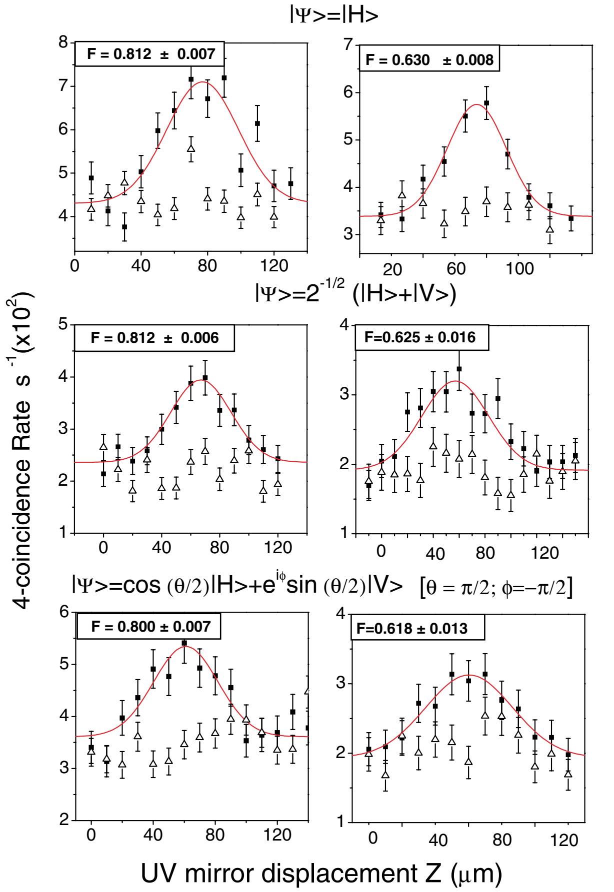

FIG. 2 (color online). Demonstration of the UOQCM (lhs) and the U- NOT gate (rhs) by 4-coincidence measurements. The values of the "fidelities" of the processes F and F^* evaluated by each test are expressed in the upper side of each plot. Filled squares: plots corresponding to "correct" polarization; Open triangles: plots corresponding to the "noise". The solid lines represent the best Gaussian fit to the results expressing the "correct" polarization.

The U-NOT gate operation has been demonstrated by restoring the PBS_2 on the mode k_2, the "anticloned" mode (AC), coupled to the detectors D_2 and D_{2^*}, via the WP_2, as shown in Fig. 1. The $\vec{\pi}$ analyzer consisting of $(PBS_2 + WP_2)$ was set as to reproduce the same filtering action of the analyzer $(PBS_T + WP_T)$ for the "trigger" signal. The devices PBS_{1a} and PBS_{1b} were removed on the C channel and the field was simply coupled by BS_1 to the detectors D_a and D_b. A coincidence event involving these ones was the signature for a cloning event. The values of the S/N ratio R^* measured by 4-coincidence experiments involving the sets $[D_2, D_T, D_a, D_b]$ and $[D_{2^*}, D_T, D_a, D_b]$ and reported in the rhs column in Fig. 2, were adopted to determine the values of the U-NOT fidelity F^*. The results are as follows: $F^*_H = 0.630 \pm 0.008$; $F^*_{H+V} = 0.625 \pm 0.016$; $F^*_{\text{left}} = 0.618 \pm 0.013$ to be compared with the optimal value: $F^* = 2/3 \approx 0.666$ [11]. Note that all results reported in Fig. 2 show an amplification efficiency which is almost identical for the different input qubits: $|\Psi\rangle_{\text{in}}$. This significant result represents the first demonstration of the universality of the QIOPA system carried out by quantum injection of a single photon qubit [4,10].

As said, the intriguing result of the present Letter is that both quantum cloning and U-NOT processes are realized optimally and contextually by the same physical apparatus and by the *same* unitary transformation over the two entangled components of a bipartite spin-$\frac{1}{2}$ space $H_1 \otimes H_2$. To the best of our knowledge it is not well understood yet why these forbidden processes can be so closely related. We may try to enlighten here at least one formal aspect of this correlation.

Note first that the overall output vector state $|\Psi\rangle_{\text{out}}$ expressed by Eq. (2) is a pure state since the unitary $\hat{U}$ acts on a pure input state. Consequently, the reduced density matrices ρ_1 and ρ_2 have the same eigenvalues and the entanglement of the bipartite state $|\Psi\rangle_{\text{out}}$ is conveniently measured by the entanglement entropy: $E(\Psi) = S(\rho_1) = S(\rho_2)$ being $S(\rho_j) \equiv -\text{Tr}\rho_j \log_2 \rho_j$ the Von Neumann entropy of either the (C) or (AC) subsystem, $i = 1, 2$ [12]. We may comment on this result by considering first the approximate cloning process performed by the UOQCM acting on the C channel, k_1. What has been actually realized in the experiment was a procedure of linearization of the cloning map which is nonlinear and as such cannot be realized exactly by nature [1,6]. By this procedure a mixed-state condition of the output state ρ_1 was achieved corresponding to the optimal limit value of the entropy $S(\rho_1) > 0$. Owing to the above expression of $E(\Psi)$, the same degree of mixedness also affects the output state realized on the AC channel, thus verifying the equation $R = R^*$ affecting the results shown in Fig. 2. Since on the AC channel an approximate CP map is realized which is generally distinct from any process realized on the C channel, e.g., here the cloning process, the above entropy equation appears to establish a significant symmetry condition in the context of axiomatic quantum theory [1]. Remarkably enough, it has been noted recently that the transformation connecting the cloning and U-NOT processes also realizes contextually the optimal entangling process and the universal probabilistic quantum processor [13].

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[11] In order to avoid spurious effects due to different *QE*s of the *D*s, the values of R were actually determined by the "amplification enhancement" shown by each "signal" plot in Fig. 2. Indeed note that the height of all coincidence signals in Fig. 2 does not decrease towards zero, as expected for large values of $|\mathbf{Z}|$, but rather towards the "noise" value C_2. This effect, due the limited time resolution of the coincidence apparatus would disappear if the resolution could be in the subpicosecond range, a hardly attainable task. In our case the t resolution was $\tau = 3$ nsec.

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