# Optimal Measurements for Simultaneous Quantum Estimation of Multiple Phases

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A quantum theory of multiphase estimation is crucial for quantum-enhanced sensing and imaging and may link quantum metrology to more complex quantum computation and communication protocols. In this Letter, we tackle one of the key difficulties of multiphase estimation: obtaining a measurement which saturates the fundamental sensitivity bounds. We derive necessary and sufficient conditions for projective measurements acting on pure states to saturate the ultimate theoretical bound on precision given by the quantum Fisher information matrix. We apply our theory to the specific example of interferometric phase estimation using photon number measurements, a convenient choice in the laboratory. Our results thus introduce concepts and methods relevant to the future theoretical and experimental development of multiparameter estimation.

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*Introduction.*—Quantum metrology is currently attracting considerable interest in light of its technological applications. Theoretical developments and experimental investigations have, so far, mostly focused on the estimation of a single phase [1–3]. Several proof-of-principle experiments have demonstrated phase estimation below the classical (shot-noise) limit [2], including applications in magnetometry [4], atomic clocks [5], and optical detection of gravitational waves [6]. Theoretical studies have focused on understanding the connection between enhancement in phase estimation and particle entanglement [7–9], as well as the impact of noise and dissipation on the fundamental bounds [10,11].

Yet, a significant class of problems cannot be efficiently cast as the estimation of a single parameter, as for quantum sensing and imaging [12] and for quantum communication and computation protocols [13,14]. Such multiparameter cases have been the subject of recent efforts, investigating the role of entanglement [15–19], and the impact of noise and decoherence [20]. Explicit examples have been considered, including measurement strategies for state estimation [21,22], the joint estimation of phase and loss rate [23,24], phase and phase diffusion [25–28], components of a displacement in phase space [29,30], multiple phases [15,18,31–34], unitary processes [35], parameters belonging to multidimensional fields [36], and estimation tasks with partial knowledge on the measurement device [37].

One of the most urgent questions in multiparameter estimation is to find saturable lower bounds of phase sensitivity. There exists a fundamental bound—the quantum Cramér-Rao bound (QCRB)—that has been formulated in Ref. [38] for the multiparameter case. However, while in the single-parameter case the QCRB is always saturable and recipes for finding the optimal measurement are known [39,40], this is not the case in the multiparameter scenario [38,41,42]. This strongly limits the possible applications of the bound.

In this Letter, we discuss the general properties that a projective measurement must have in order to saturate the QCRB for multiphase estimation with probe pure states. Our results extend and complement previous works [31,38,42] by identifying the necessary and sufficient conditions on the projectors and constructing optimal measurements. They apply also when the generators of phase-encoding transformations do not commute. The restriction to the ideal case of pure states may provide a guideline for experiments implementing close-to-pure states with high fidelity as, for instance, multiarm interferometry with single photons in each input port [43,44]. Furthermore, projective measurements are known to be optimal in the single-parameter case [39], and previous studies have found examples of measurements saturating the QCRB within this class [15,31]. In particular, our results apply to the detection of the number of particles at the output ports of an interferometer, which realizes in the ideal lossless scenario a projective measurement. Overall, our findings provide an effective guideline to design and optimize an experimental apparatus aiming at multiparameter estimation. This has implications for future quantum technology developments with photons, atoms, and trapped ions.

*Multiphase estimation.*—We consider the simultaneous estimation of a $d$-dimensional vector parameter $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_d\}$ using separable measurements and following four steps (see Fig. 1). (i) A probe pure state $|\psi\rangle$ is prepared. (ii) It is shifted in phase by applying a phase-encoding unitary transformation $\hat{U}(\boldsymbol{\theta})$. (iii) The output state

 

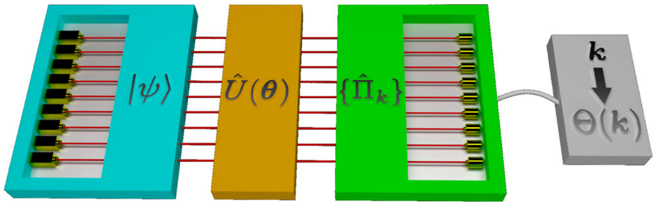


FIG. 1. General framework of multiparameter estimation considered in this Letter: A probe state $|\psi\rangle$ is prepared, transformed according to the unitary parameter-dependent transformation $\hat{U}(\boldsymbol{\theta})$, and measured by a set of projectors $\{\hat{\Pi}_k\}$. The vector parameter is retrieved via an estimator $\Theta(\boldsymbol{k})$, which is a function of the measurement outcomes.

$|\psi_{\boldsymbol{\theta}}\rangle \equiv \hat{U}(\boldsymbol{\theta})|\psi\rangle$ is detected. In the following, we consider a set of projective measurements $\{\hat{\Pi}_k\}$, labeled by the index $k$ representing the possible result. Eventually, the protocol is repeated $\nu$ times using independent and identical copies of the output state (with the same transformation and output measurement). (iv) Finally, from the output results $\boldsymbol{k} = \{k_1, \ldots, k_\nu\}$, one infers the vector parameter via an estimator $\Theta(\boldsymbol{k})$. The probability of observing the sequence $\boldsymbol{k}$, conditioned to the vector parameter $\boldsymbol{\theta}$, is given by $P(\boldsymbol{k}|\boldsymbol{\theta}) = \prod_{i=1}^{\nu} P(k_i|\boldsymbol{\theta})$ and $P(k|\boldsymbol{\theta}) = \langle\psi_{\boldsymbol{\theta}}|\hat{\Pi}_k|\psi_{\boldsymbol{\theta}}\rangle$. In the following, we consider locally unbiased estimators, for which the average value of the estimator equals the true value of the parameter: $\bar{\Theta}(\boldsymbol{k}) = \sum_{\boldsymbol{k}} P(\boldsymbol{k}|\boldsymbol{\theta})\Theta(\boldsymbol{k}) = \boldsymbol{\theta}$ and $d\bar{\Theta}(\boldsymbol{k})/d\theta_l = 1$ $(l = 1, \ldots, d)$. A figure of merit of the phase sensitivity is the covariance matrix,

$$[\boldsymbol{B}(\boldsymbol{\theta})]_{l,m} = \sum_{\boldsymbol{k}} P(\boldsymbol{k}|\boldsymbol{\theta})[\Theta(\boldsymbol{k}) - \boldsymbol{\theta}]_l[\Theta(\boldsymbol{k}) - \boldsymbol{\theta}]_m. \quad (1)$$

The diagonal elements $\boldsymbol{B}_{l,l}$ equal the variance $(\Delta\theta_l)^2$. For any unbiased estimators and independent measurements, the chain of inequalities

$$\boldsymbol{B}(\boldsymbol{\theta}) \geq \frac{[\boldsymbol{F}(\boldsymbol{\theta})]^{-1}}{\nu} \geq \frac{[\boldsymbol{F}_Q(\boldsymbol{\theta})]^{-1}}{\nu} \quad (2)$$

holds (in the matrix sense). The first inequality is the Cramér-Rao bound, where

$$[\boldsymbol{F}(\boldsymbol{\theta})]_{l,m} = \sum_k \frac{\partial_l P(k|\boldsymbol{\theta})\partial_m P(k|\boldsymbol{\theta})}{P(k|\boldsymbol{\theta})} \quad (3)$$

is the $d \times d$ symmetric Fisher information matrix (FIM) and $\partial_l = \partial/\partial\theta_l$. Since $\partial_l P(k|\boldsymbol{\theta}) = 2\mathrm{Re}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\hat{\Pi}_k|\psi_{\boldsymbol{\theta}}\rangle]$, the FIM depends on how the measurement set acts on the Hilbert subspace $\mathcal{H}$ spanned by the probe state $|\psi_{\boldsymbol{\theta}}\rangle$ and $\{|\partial_l\psi_{\boldsymbol{\theta}}\rangle\}_{l=1,\ldots,d}$. Here Re$[x]$ and Im$[x]$ indicate the real and imaginary part of $x$, respectively, and $|\partial_l\psi_{\boldsymbol{\theta}}\rangle = -i\hat{H}_l|\psi_{\boldsymbol{\theta}}\rangle$, where $\hat{H}_l \equiv i[\partial_l\hat{U}(\boldsymbol{\theta})]\hat{U}^\dagger(\boldsymbol{\theta})$ is a Hermitian operator. The Cramér-Rao bound can be established only if the FIM is strictly positive (and thus invertible). In this case, it can always be saturated, asymptotically in $\nu$, by the maximum likelihood estimator [38]. The second bound in Eq. (2) is due to $\boldsymbol{F}(\boldsymbol{\theta}) \leq \boldsymbol{F}_Q(\boldsymbol{\theta})$ [38], where $\boldsymbol{F}_Q(\boldsymbol{\theta})$ is the quantum Fisher information matrix (QFIM),

$$[\boldsymbol{F}_Q(\boldsymbol{\theta})]_{l,m} = 4\mathrm{Re}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\partial_m\psi_{\boldsymbol{\theta}}\rangle] + 4\langle\partial_l\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle\langle\partial_m\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle. \quad (4)$$

We recall that the inequality $\boldsymbol{F} \leq \boldsymbol{F}_Q$ is understood in the matrix sense; i.e., $\boldsymbol{u}^\top\boldsymbol{F}\boldsymbol{u} \leq \boldsymbol{u}^\top\boldsymbol{F}_Q\boldsymbol{u}$ holds for arbitrary $d$-dimensional real vectors $\boldsymbol{u}$. The inverse of the QFIM—when it exists—sets a lower bound (2) for the simultaneous estimation of multiple parameters, called the QCRB, which depends only on the probe state and phase-encoding transformation. The QFIM allows you to assess, for a given phase-encoding transformation, the intrinsic merit of a particular probe state, optimized over all measurements. There is a major problem though: While in the single parameter case it is always possible to choose an optimal measurement for which the equality $\boldsymbol{F} = \boldsymbol{F}_Q$ holds [39,40], in the multiparameter case $(d > 1)$ such an optimal measurement does not exist, in general. The search for conditions under which the FIM saturates the QFIM has long engaged the field of quantum metrology [38,42]. For noncommuting operators, the main result available in the literature is due to Matsumoto [42].

*Weak commutativity theorem.*—Given the pure state $|\psi_{\boldsymbol{\theta}}\rangle$, it is possible to saturate $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F}_Q(\boldsymbol{\theta})$ if and only if

$$\mathrm{Im}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\partial_m\psi_{\boldsymbol{\theta}}\rangle] = 0, \quad \forall\ l, m. \quad (5)$$

To be more precise, the condition (5) is necessary, and, if Eq. (5) holds and the QFIM is invertible, it is possible to construct a set of projectors for which $\boldsymbol{F} = \boldsymbol{F}_Q$ holds [42]. Note that the (strong) commutativity condition $[\hat{H}_l, \hat{H}_m] = 0$ for all $l$, $m$ implies Eq. (5), while $\boldsymbol{F} = \boldsymbol{F}_Q$ is also possible for noncommuting generators [36,42].

In general, any experimental apparatus is set by a specific probe, transformation, and final measurement, often taken to be projective. The saturation of the QCRB indicates the optimal metrological performance, and it is desirable to know whether there exist configurations of the apparatus for which $\boldsymbol{F} = \boldsymbol{F}_Q$. The saturation can be inspected explicitly (with a calculation of the FIM and the QFIM); however, this may be technically demanding, and failing at saturating would not give guidance on how to improve the measurement. Furthermore, the weak commutativity theorem provides specific measurements for which $\boldsymbol{F} = \boldsymbol{F}_Q$ holds that might not be those implemented experimentally.

In the following, we provide three theorems giving necessary and sufficient conditions on projective measurements to saturate $\boldsymbol{F} = \boldsymbol{F_Q}$ for an arbitrary generator of phase encoding. These theorems indicate the full class of projective measurements saturating the QFIM for pure states. The knowledge of these optimal projectors provides a guideline for designing future multiparameter estimation experiments. It is worth pointing out that, if the FIM is invertible, then our conditions become necessary and sufficient to saturate the QCRB with projective measurements. We discuss the main consequences of our findings and the relation with existing results. The proof of the theorems is detailed in Supplemental Material [45].

*Theorem 1 (projective measurement orthogonal to the probe).*—Given a pure state $|\psi_{\boldsymbol{\theta}}\rangle$ and a set of projectors $\{|\Upsilon_k\rangle\langle\Upsilon_k|\}$ on the state itself ($|\Upsilon_1\rangle = |\psi_{\boldsymbol{\theta}}\rangle$, for $k=1$) and on the orthogonal subspace ($\langle\Upsilon_k|\psi_{\boldsymbol{\theta}}\rangle = 0$, for $k \neq 1$), $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ holds if and only if

$$\lim_{\boldsymbol{\varphi}\to\boldsymbol{\theta}} \frac{\mathrm{Im}[\langle\partial_l\psi_{\boldsymbol{\varphi}}|\Upsilon_k\rangle\langle\Upsilon_k|\psi_{\boldsymbol{\varphi}}\rangle]}{|\langle\Upsilon_k|\psi_{\boldsymbol{\varphi}}\rangle|} = 0, \tag{6}$$

for all $l = 1, 2, \ldots, d$ and all $k \neq 1$.

It is possible to demonstrate that Eq. (6) is consistent with Eq. (5) [45]. In the single-parameter case ($d = 1$), where $\hat{U}(\theta) = e^{-i\hat{H}\theta}$ and $\hat{H}$ is a Hermitian operator, Eq. (6) is always satisfied [45,46]. In Ref. [31], it has been claimed that any projective measurement orthogonal to the probe state saturates the QFIM in the multiparameter case ($d > 1$). We emphasize that only the projectors that satisfy Eq. (6) saturate the QFIM. The condition (6) is highly nontrivial: It is easy to find projective measurements that do not fulfill Eq. (6), and for which $\boldsymbol{F} \neq \boldsymbol{F_Q}$; see the examples at the end of this Letter.

We can perform the limit in Eq. (6) under the assumption that $\langle\Upsilon_k|\partial_j\psi_{\boldsymbol{\theta}}\rangle = 0$ does not hold for all $j = 1, \ldots, d$ [45]. In this case, we have $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ if and only if

$$\mathrm{Im}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\Upsilon_k\rangle\langle\Upsilon_k|\partial_m\psi_{\boldsymbol{\theta}}\rangle] = 0, \tag{7}$$

for all indices $l, m = 1, 2, \ldots, d$ and all $k$. If $\langle\Upsilon_k|\partial_j\psi_{\boldsymbol{\theta}}\rangle = 0$ for all $j = 1, \ldots, d$, then it is possible to find conditions similar to Eq. (7) and involving higher-order derivatives [45]. A consequence of the above theorem is the following.

*Corollary 1.*—Given a probe pure state and unitary transformations satisfying Eq. (5), it is always possible to saturate $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ by a set of projectors given by the probe state itself and a suitable choice of vectors on the orthogonal subspace.

Here we explicitly construct optimal projectors saturating the QFIM. We recall that $\boldsymbol{F}(\boldsymbol{\theta})$ depends only on $|\psi_{\boldsymbol{\theta}}\rangle$ and the $d$ vectors $\{|\partial_m\psi_{\boldsymbol{\theta}}\rangle\}_{m=1,\ldots,d}$. In general, the states $|\partial_m\psi_{\boldsymbol{\theta}}\rangle$ are not orthogonal to the probe. To construct a set of projectors orthogonal to $|\psi_{\boldsymbol{\theta}}\rangle$, we introduce the set of states $|\omega_m\rangle \equiv |\partial_m\psi_{\boldsymbol{\theta}}\rangle + |\psi_{\boldsymbol{\theta}}\rangle\langle\partial_m\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle$, for $m = 1, \ldots, d$ [47]. These satisfy $\langle\psi_{\boldsymbol{\theta}}|\omega_m\rangle = 2\mathrm{Re}[\langle\partial_m\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle] = \partial_m\langle\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle = 0$. The states $|\omega_m\rangle$ are not orthogonal to each other, in general, and we can introduce the $d \times d$ Gram matrix $\boldsymbol{\Omega}_{l,m} = \langle\omega_l|\omega_m\rangle = \langle\partial_l\psi_{\boldsymbol{\theta}}|\partial_m\psi_{\boldsymbol{\theta}}\rangle + \langle\partial_l\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle\langle\partial_m\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle$. It should be noticed that, according to Eq. (5), the saturation of the QFIM requires $\mathrm{Im}[\boldsymbol{\Omega}] = \mathrm{Im}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\partial_m\psi_{\boldsymbol{\theta}}\rangle] = 0$. We thus necessarily restrict to matrices $\boldsymbol{\Omega}$ that are real and symmetric. In particular, $\boldsymbol{F_Q} = 4\boldsymbol{\Omega}$ [see Eq. (4)], and the QFIM is positive definite (and thus invertible) if and only if the states $|\omega_m\rangle$ are linearly independent. We can construct via the Gram-Schmidt process an orthogonal basis of the subspace $\mathcal{H}\backslash|\psi_{\boldsymbol{\theta}}\rangle\langle\psi_{\boldsymbol{\theta}}|$ as linear combinations $|\Upsilon_k\rangle = \sum_{m=1}^d b_{m,k}|\omega_m\rangle$, with real coefficients $b_{m,k}$. With this choice, Eq. (7) is satisfied by the constructed set $\{|\Upsilon_k\rangle\}$, since $\langle\partial_l\psi_{\boldsymbol{\theta}}|\Upsilon_k\rangle = \sum_{m=1}^d \boldsymbol{\Omega}_{l,m} b_{m,k}$ is real. This concludes the proof of the corollary.

*Theorem 2 (projective measurement not orthogonal to the probe).*—Given a probe pure state $|\psi_{\boldsymbol{\theta}}\rangle$ and a set of projectors $\{|\Upsilon_k\rangle\langle\Upsilon_k|\}$ not orthogonal to the probe (i.e., $\langle\Upsilon_k|\psi_{\boldsymbol{\theta}}\rangle \neq 0$ for all $k$), $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ holds if and only if

$$\mathrm{Im}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\Upsilon_k\rangle\langle\Upsilon_k|\psi_{\boldsymbol{\theta}}\rangle] = |\langle\psi_{\boldsymbol{\theta}}|\Upsilon_k\rangle|^2\mathrm{Im}[\langle\partial_l\psi_{\boldsymbol{\theta}}|\psi_{\boldsymbol{\theta}}\rangle], \tag{8}$$

for all $l = 1, 2, \ldots, d$ and all $k$.

In the single-parameter case, Eq. (8) becomes $\mathrm{Re}[\langle\psi_{\boldsymbol{\theta}}|\hat{H}|\Upsilon_k\rangle\langle\Upsilon_k|\psi_{\boldsymbol{\theta}}\rangle] = |\langle\psi_{\boldsymbol{\theta}}|\Upsilon_k\rangle|^2\langle\psi_{\boldsymbol{\theta}}|\hat{H}|\psi_{\boldsymbol{\theta}}\rangle$, for all $\Upsilon_k$. This is precisely the necessary and sufficient conditions given in Ref. [48] for the saturation of the quantum Fisher information for a single parameter. It is also possible to demonstrate that Eq. (8) implies the weak commutativity condition (5); see [45].

*Corollary 2.*—Given a pure probe state and unitary transformations satisfying Eq. (5), there always exists a set of projectors nonorthogonal to the probe which saturates $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$.

We prove the corollary by constructing a set of projectors that satisfy Eq. (8). Restricting to the subspace $\mathcal{H}$, we can decompose the states $|\Upsilon_k\rangle$ in the basis given by $|\psi_{\boldsymbol{\theta}}\rangle$ and $\{|\partial_m\psi_{\boldsymbol{\theta}}\rangle\}_{m=1,\ldots,d}$. We take $|\Upsilon_k\rangle = \sum_{m=1}^d b_{m,k}|\omega_m\rangle + b_{d+1,k}|\psi_{\boldsymbol{\theta}}\rangle$, where we require $b_{d+1,k} = \langle\psi_{\boldsymbol{\theta}}|\Upsilon_k\rangle \neq 0$ for all $k$. Equation (8) is fulfilled by taking real $b_{m,k}$ (for $m = 1, \ldots, d+1$) and noticing that $\langle\partial_l\psi_{\boldsymbol{\theta}}|\omega_n\rangle$ is necessarily real to fulfill Eq. (5) [49]. The real coefficients $b_{m,k}$ must be chosen such that $\sum_k|\Upsilon_k\rangle\langle\Upsilon_k| = \mathbb{1}$. An orthonormal set that fulfills all these conditions can be constructed via the Gram-Schmidt process.

*Theorem 3 (general projective measurement).*—Consider a probe pure state $|\psi_{\boldsymbol{\theta}}\rangle$ and a set of projectors $\{|\Upsilon_k\rangle\langle\Upsilon_k|\}$. $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ holds if and only if Eq. (6) is fulfilled for all indices $l$, $m$ and all $k$ for which $\langle\Upsilon_k|\psi_{\boldsymbol{\theta}}\rangle = 0$ and Eq. (8) is fulfilled for all indices $l$ and all $k$ for which $\langle\Upsilon_k|\psi_{\boldsymbol{\theta}}\rangle \neq 0$.

*Corollary 3.*—Given a pure probe state and unitary transformations satisfying Eq. (5), it is always possible to find a set of projectors satisfying Theorem 3.

The proof of this statement immediately follows from Corollary 2 by taking real coefficients $b_{m,k}$ (for $m = 1, \ldots, d$) and $b_{d+1,k}$ either real and finite or equal to zero. We point out that the projective measurement previously constructed by Matsumoto [42] to saturate the equality $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ explicitly requires $\boldsymbol{F_Q}(\boldsymbol{\theta})$ to be invertible, a condition which is not general and not required in our case. It is possible to show that the specific class of projective measurement used in the proof of the weak commutativity theorem satisfies the conditions of Theorem 3.

*Examples.*—In the following, we illustrate our theoretical results with experimentally relevant examples. Using our theorems, we predict the saturation of $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ and demonstrate that it is consistent with a direct calculation of the FIM and the QFIM.

Let us consider an $n$-mode Mach-Zehnder interferometer (MZI) that allows the simultaneous estimation of multiple (up to $n-1$) phases. Three- and four-mode MZIs are currently within the reach of present technology [43,44,50]. We first discuss the three-mode case with a Fock state $|\psi\rangle = |1,1,1\rangle$ as input. The initial step of the interferometer is a three-mode splitter, a tritter, described by the unitary matrix $U^{(3)}_{j,k} = 3^{-1/2} e^{i2\pi/3(1-\delta_{j,k})}$. We consider the estimation of the phase difference $\theta_1 = \phi_1 - \phi_3$ and $\theta_2 = \phi_2 - \phi_3$, where $\phi_1$, $\phi_2$, and $\phi_3$ are the shifts of each interferometric mode. After phase encoding, the modes recombine at a second tritter described by a unitary matrix $[U^{(3)}]^{-1}$ [45]. Finally, photons in each mode are counted. This measurement corresponds to a projection over all the possible states $|\Upsilon_k\rangle = |i,j,h\rangle$ with $i+j+h = 3$. We test the conditions (6) and (8) for different values of $\boldsymbol{\theta} = (\theta_1, \theta_2)$. In particular, for $\boldsymbol{\theta} = (0,0)$, we have a projection over the probe state and over the orthogonal subspace. We find that the conditions (6) and (8) are not fulfilled. In order to verify the results of our theorem, we calculate explicitly $\boldsymbol{F}$ and $\boldsymbol{F_Q}$ and plot, in Fig. 2(a), $\|\boldsymbol{F} - \boldsymbol{F_Q}\|_2$ by varying $\boldsymbol{\theta}$ (the norm $\|\boldsymbol{F} - \boldsymbol{F_Q}\|_2$ ranges from 0 to $\|\boldsymbol{F_Q}\|_2$ that is equal to 8 in this case [45]). We find that $\boldsymbol{F}(\boldsymbol{\theta}) \neq \boldsymbol{F_Q}(\boldsymbol{\theta})$ ($\|\boldsymbol{F} - \boldsymbol{F_Q}\|_2 > 3/4$, in particular), consistent with the predictions of our theorems. We have repeated the analysis for a four-mode interferometer, consisting of two cascaded quarters, with a four-photon $|1,1,1,1\rangle$ Fock state as input. The quarters are optical devices represented by the unitary matrix $U^{(4)}_{j,k} = 2^{-1}(-1)^{1-\delta_{j,k}}$. To directly compare with the three-mode case and for the sake of illustration, we consider again the estimation of two phases [51] and choose photon-counting measurement. In this case, for certain values of $\boldsymbol{\theta}$ the conditions (6) and (8) are fulfilled, and the QFIM is saturated for the estimation of the two phases simultaneously [see Fig. 2(b)]: These are the values $\theta_1 = \theta_2 \in [0,\pi]$ (dashed line) and the points $\boldsymbol{\theta} = (0,\pi)$ and $\boldsymbol{\theta} = (\pi, 0)$ (circles). Consistently, by a direct calculation we find that $\|\boldsymbol{F} - \boldsymbol{F_Q}\|_2 = 0$ for these values of $\boldsymbol{\theta}$. This result can be generalized by considering that multiport symmetric beam splitters can be constructed from Sylvester matrices, which can be defined in dimension $n = 2^m$ (with $m$ an integer) and have all real coefficients equal to $\pm 1$ [52]. By employing an $n$-photon Fock state as the probe and photon counting as the measurement, the QFIM for the simultaneous estimation of $d = n-1$ phases can always be saturated by checking condition (7) when $\theta_i = 0$ for $i = 1, \ldots, d$.

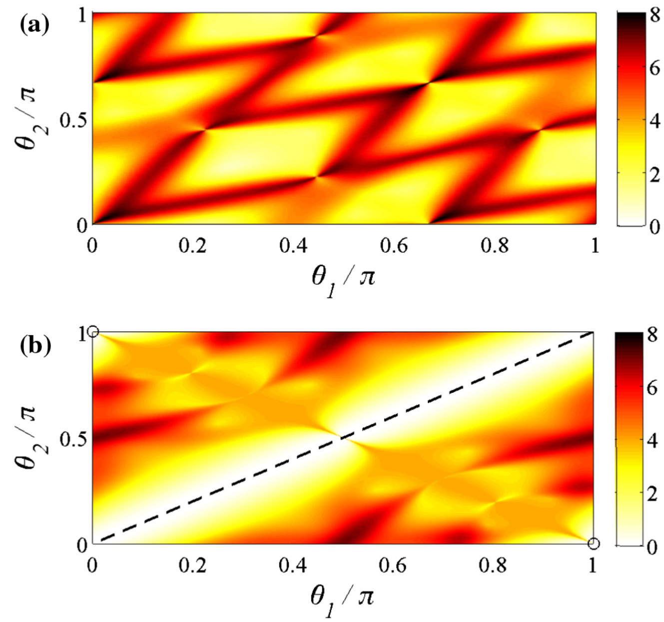


FIG. 2. $\|\boldsymbol{F} - \boldsymbol{F_Q}\|_2$ of a three-mode (a) and a four-mode (b) interferometer (see the text and Ref. [45]) for the estimation of $\boldsymbol{\theta} = (\theta_1, \theta_2)$. For the three-mode interferometer, there is no value of $\boldsymbol{\theta}$ where the theorems are satisfied. Consistently, we find $\|\boldsymbol{F} - \boldsymbol{F_Q}\|_2 > 3/4$. For the four-mode interferometer, we can find values of $\boldsymbol{\theta}$ where the saturation conditions $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F_Q}(\boldsymbol{\theta})$ discussed in the Letter are fulfilled: These are given by the dashed line and circles.

*Conclusions.*—We have provided necessary and sufficient conditions on projective measurements to saturate the quantum Fisher information matrix in the case of pure probe states. We have also shown how to construct such optimal projectors and have tested the theory for experimentally relevant configurations. Finally, we recall that our conditions become necessary and sufficient for the Cramér-Rao bound to saturate the quantum Cramér-Rao bound when the quantum Fisher information matrix is invertible. Our results are a step forward to the theoretical understanding of multiparameter estimation and a key for the experimental design of future quantum imaging and multiparameter metrology devices.

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