

Fully nonlocal quantum correlations

Leandro Aolita,¹ Rodrigo Gallego,¹ Antonio Acín,^{1,2} Andrea Chiuri,³ Giuseppe Vallone,^{3,4}
Paolo Mataloni,^{3,5} and Adán Cabello^{6,7}

¹*Institut de Ciències Fotòniques, E-08860 Castelldefels, Barcelona, Spain*

²*Institució Catalana de Recerca i Estudis Avançats, Lluís Companys 23, E-08010 Barcelona, Spain*

³*Dipartimento di Fisica, Università Sapienza di Roma, I-00185 Roma, Italy*

⁴*Museo Storico della Fisica e Centro Studi e Ricerche Enrico Fermi, Via Panisperna 89/A, Compendio del Viminale, I-00184 Roma, Italy*

⁵*Istituto Nazionale di Ottica, CNR, Largo E. Fermi 6, I-50125 Florence, Italy*

⁶*Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain*

⁷*Department of Physics, Stockholm University, S-10691 Stockholm, Sweden*

(Received 18 May 2011; revised manuscript received 24 October 2011; published 5 March 2012)

Quantum mechanics is a nonlocal theory, but not as nonlocal as the no-signalling principle allows. However, there exist quantum correlations that exhibit maximal nonlocality: they are as nonlocal as any nonsignalling correlation and thus have a local content, quantified by the fraction p_L of events admitting a local description, equal to zero. We exploit the known link between the Kochen-Specker and Bell theorems to derive a maximal violation of a Bell inequality from every Kochen-Specker proof. We then show that these Bell inequalities lead to experimental bounds on the local content of quantum correlations that are significantly better than those based on other constructions. We perform the experimental demonstration of a Bell test originating from the Peres-Mermin Kochen-Specker proof, providing an upper bound on the local content $p_L \lesssim 0.22$.

DOI: [10.1103/PhysRevA.85.032107](https://doi.org/10.1103/PhysRevA.85.032107)

PACS number(s): 03.65.Ud, 03.67.Mn, 42.50.Xa

I. INTRODUCTION

Since the seminal work by Bell [1], we know that there exist quantum correlations that cannot be thought of classically. This impossibility is known as nonlocality and follows from the fact that the correlations obtained when performing local measurements on entangled quantum states may violate a Bell inequality, which sets conditions satisfied by all classically correlated systems.

The standard nonlocality scenario consists of two distant systems on which two observers, Alice and Bob, perform respectively m_a and m_b different measurements of d_a and d_b possible outcomes. The outcomes of Alice and Bob are respectively labeled a and b , while their measurement choices are x and y , with $a = 1, \dots, d_a$, $b = 1, \dots, d_b$, $x = 1, \dots, m_a$, and $y = 1, \dots, m_b$. The correlations between the two systems are encapsulated in the joint conditional probability distribution $P(a, b|x, y)$.

This probability distribution should satisfy the no-signalling principle, which states that no faster-than-light communication is possible. When the measurements by the two observers define spacelike separated events, this implies that the marginal distributions for Alice (Bob) should not depend on Bob's (Alice's) measurement choice, i.e., $\sum_b P(a, b|x, y) = P(a|x)$, $\forall y$, and similarly for Bob. These linear constraints define the set of nonsignalling correlations. Quantum correlations in turn are those that can be written as $P(a, b|x, y) = \text{tr}(\rho_{AB} M_a^x \otimes M_b^y)$, where ρ_{AB} is a bipartite quantum state and M_a^x and M_b^y define local measurements by the observers. Finally, classical correlations are defined as those that can be written as $P(a, b|x, y) = \sum_\lambda p(\lambda) P_A(a|x, \lambda) P_B(b|y, \lambda)$. These correlations are also called local, as outcome a (b) is locally generated from input x (y) and the preestablished classical correlations λ .

The violation of Bell inequalities by entangled states implies that the set of quantum correlations is strictly larger

than the classical one. A similar gap appears when considering quantum versus general nonsignalling correlations: there exist correlations that, despite being compatible with the no-signalling principle, cannot be obtained by performing local measurements on any quantum system [2]. In particular, there exist nonsignalling correlations that exhibit stronger nonlocality, in the sense that they give larger Bell violations, than any quantum correlations [see Fig. 1(a)].

Interestingly, there are situations in which this second gap disappears: quantum correlations are then maximally nonlocal, as they are able to attain the maximal Bell violation compatible with the no-signalling principle. Geometrically, in these extremal situations quantum correlations reach the border of the set of nonsignalling correlations [see Fig. 1(b)]. From a quantitative point of view, it is possible to detect this effect by computing the local fraction [3] of the correlations. This quantity measures the fraction of events that can be described by a local model. Given $P(a, b|x, y)$, consider all possible decompositions,

$$P(a, b|x, y) = q_L P_L(a, b|x, y) + (1 - q_L) P_{NL}(a, b|x, y), \quad (1)$$

in terms of arbitrary local and nonsignalling distributions, $P_L(a, b|x, y)$ and $P_{NL}(a, b|x, y)$, with respective weights q_L and $1 - q_L$, where $0 \leq q_L \leq 1$. The local fraction of $P(a, b|x, y)$ is defined as the maximum local weight over all possible decompositions as (1):

$$p_L \doteq \max_{\{P_L, P_{NL}\}} q_L. \quad (2)$$

It can be understood as a measure of the nonlocality of the correlations. Maximally nonlocal correlations feature $p_L = 0$ [see Fig. 1(b)].

Any Bell violation provides an upper bound on the local fraction of the correlations that cause it. In fact, a Bell inequality is defined as $\sum T_{a,b,x,y} P(a, b|x, y) \leq \beta_L$, where $T_{a,b,x,y}$ is a tensor of real coefficients. The maximal value of

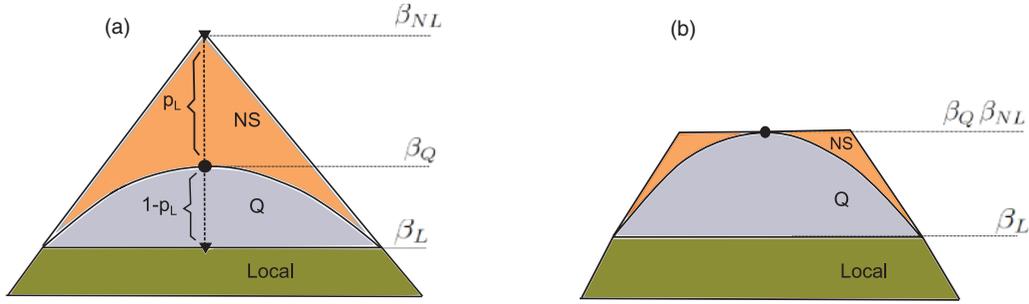


FIG. 1. (Color online) Nonsignalling, quantum, and classical correlations. The set of nonsignalling correlations defines a polytope. The set of quantum correlations is contained in the set of nonsignalling correlations. The set of classical correlations is also a polytope and is contained inside the quantum set. (a) In general, the set of quantum correlations is not tangent to the set of nonsignalling correlations. This means that the maximal value β_Q of a Bell inequality achievable by quantum correlations is above the local bound β_L but strictly smaller than the maximal value β_{NS} for nonsignalling correlations. (b) In the present Bell tests, in contrast, quantum correlations are tangent to the set of nonsignalling correlations and thus attain the nonsignalling value of a Bell inequality. The corresponding upper bound on the local fraction is zero, which discloses the full nonlocal nature of quantum mechanics.

the left-hand side of this inequality over classical correlations defines the local bound β_L , whereas its maximum over quantum and nonsignalling correlations gives the maximal quantum and nonsignalling values β_Q and β_{NL} , respectively. From this and (1) it follows immediately that [4]

$$p_L \leq \frac{\beta_{NL} - \beta_Q}{\beta_{NL} - \beta_L} \doteq p_{L_{\max}}. \quad (3)$$

Thus, quantum correlations feature $p_L = 0$ if (and, in fact, only if) they violate a Bell inequality as much as any nonsignalling correlations.

In this work we study the link between the Kochen-Specker (KS) [5] and Bell's theorems, previously considered in Refs. [6–10]. We recast this link in the form of Bell inequalities maximally violated by quantum states. We then show that the resulting Bell inequalities can be used to get experimental bounds on the nonlocal content of quantum correlations that are significantly better than Bell tests based on more standard Bell inequalities or multipartite Greenberger-Horne-Zeilinger (GHZ) paradoxes [11]. This allows us to perform an experimental demonstration, which yields an experimental upper bound on the local part $p_{L_{\max}} = 0.218 \pm 0.014$. To our knowledge, this represents the lowest value ever reported, even taking into account multipartite Bell tests.

II. GENERAL FORMALISM

In this section, we present the details of the construction to derive different Bell inequalities maximally violated by quantum mechanics from every proof of the KS theorem. This construction was first introduced in [6] and later was applied in the context of “all-versus-nothing” nonlocality tests [7], pseudotelepathy games (see [8] and references therein), the free-will theorem [9], and quantum key distribution [10]. Here we exploit it to generate quantum correlations with no local part.

Recall that the KS theorem studies whether deterministic outcomes can be assigned to von Neumann quantum measurements, in contrast to the quantum formalism, which can only assign probabilities. A von Neumann measurement z is defined by a set of d orthogonal projectors acting on a Hilbert space of dimension d . Consider m such measurements,

given by $m \times d$ rank-1 projectors Π_i^z , with $z = 1, \dots, m$ and $i = 1, \dots, d$, such that $\Pi_i^z \Pi_{i'}^z = \delta_{i,i'}$ and $\sum_i \Pi_i^z = \mathbb{1}$ for all z , with $\mathbb{1}$ being the identity operator. The theorem studies maps from these measurements to deterministic d -outcome probability distributions. In addition, an extra requirement is imposed on the maps: the assignment has to be noncontextual. That is, if a particular outcome, corresponding to a projector Π_i^z , is assigned to a given measurement, the same outcome must be assigned to all the measurements in which this projector appears. Formally, this means that the assignment map, denoted by v_A , acts actually on projectors rather than on measurements: $v_A(\Pi_i^z) \in \{0, 1\}$, such that $\sum_i v_A(\Pi_i^z) = 1$ for all z . The KS theorem shows that noncontextual deterministic assignments do not exist.

Although this impossibility follows as a corollary of Gleason's theorem [12], one virtue of the proofs of the KS theorem [5, 13–15] is that they involve a finite number of measurements. More precisely, each KS proof consists of a set of m measurements (contexts) as above but chosen so as to share altogether p projectors $\tilde{\Pi}_j$, with $j = 1, \dots, p$, that make noncontextual deterministic assignments incompatible with the measurements' structure. Denote by D_j the set of two-tuples $D_j = \{(i, z)\}$ such that $(i, z) \in D_j$ if $\Pi_i^z = \tilde{\Pi}_j$. Each set of two-tuples D_j collects the indexes of all common projectors among all different measurements.

Let us now see how this highly nontrivial configuration of measurements can be used to derive maximally nonlocal quantum correlations. Consider the standard Bell scenario depicted in Fig. 2(b). Two distant observers (Alice and Bob) perform uncharacterized measurements in a device-independent scenario. Let us assume that Alice can choose among $m_a = m$ measurements of $d_a = d$ outcomes. On the other hand, Bob can choose among $m_b = p$ measurements of $d_b = 2$ outcomes, labeled by 0 and 1. We denote Alice's (Bob's) measurement choice by x (y) and her (his) outcome by a (b). Collecting statistics at many instances of the experiment, they compute the quantity $P(a, b|x, y)$, namely, the probability of obtaining outcome a and b when measurements x and y were performed.

Consider next the following quantum realization of the experiment: Alice and Bob perform their measurements on

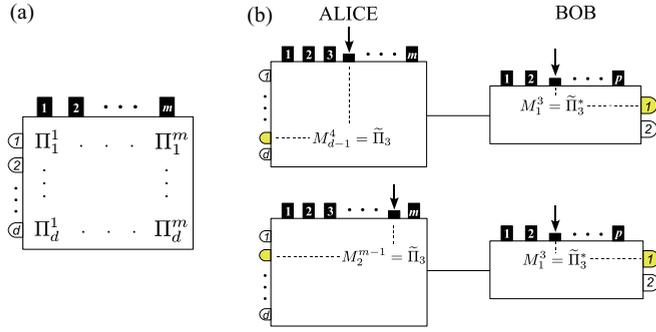


FIG. 2. (Color online) Noncontextual assignments in the black-box scenario. (a) A KS proof consists of a single observer, say Alice, who performs m measurements of d outcomes. The KS proof requires that outcomes of different measurements correspond to the same projector. There are altogether p projectors, denoted by $\tilde{\Pi}_j$, shared by different measurements. The common projectors impose constraints that, if the outcomes are assigned by noncontextual deterministic maps, lead to contradictions. (b) In the Bell test associated with the KS proof, Bob's box has $m_b = p$ possible measurements of $d_b = 2$ outcomes. In the quantum setting, the two observers share a maximally entangled state. Alice makes $m_a = m$ measurements of $d_a = d$ outcomes, which correspond to the observables in the KS proof. Bob's measurements are perfectly correlated with the p projectors $\tilde{\Pi}_j$ on Alice's side, thanks to the properties of the maximally entangled state. A local model reproducing all these correlations would imply the existence of a deterministic noncontextual model for Alice's measurements, which is impossible.

the bipartite maximally entangled state $|\psi_d\rangle = \sum_{k=0}^{d-1} \frac{1}{\sqrt{d}} |kk\rangle$. When Alice chooses input x , measurement $\{M_a^x = \Pi_i^z$, with $x = z$ and $a = i\}$ is performed. In turn, when Bob chooses input y , the following measurement takes place: $\{M_1^y = (\tilde{\Pi}_j)^*, M_2^y = \mathbb{1} - (\tilde{\Pi}_j)^*$, with $y = j\}$, where the asterisk (*) denotes complex conjugation. The properties of $|\psi_d\rangle$ guarantee that these measurements by Alice and Bob are perfectly correlated. Furthermore, they lead to the nonsignalling value β_{NS} of the following linear combination of probabilities:

$$\beta(P(a,b|x,y)) = \sum_{y=1}^p \sum_{(a',x) \in D_y} [P(a = a', b = 1|x,y) + P(a \neq a', b = 0|x,y)]. \quad (4)$$

Indeed, for all the terms appearing in (4), $P(a = a', b = 1|x,y) + P(a \neq a', b = 0|x,y) = 1$. This can be easily seen by noticing that if Bob's output is equal to 1, Alice's system is projected onto $\tilde{\Pi}_y = \Pi_a^x$, and thus, the result of Alice's measurement x is a' . On the contrary, if Bob's box outputs 0, Alice's system is projected onto $\mathbb{1} - \tilde{\Pi}_y = \mathbb{1} - \Pi_a^x$, and thus, Alice's outcome is such that $a \neq a'$. As the sum of the two probabilities $P(a = a', b = 1|x,y)$ and $P(a \neq a', b = 0|x,y)$ can never be larger than 1, one has

$$\beta_Q = \beta_{NS} \doteq \sum_{y=1}^p \sum_{(a',x) \in D_y} 1. \quad (5)$$

As for local correlations, we now show that it is $\beta_L \leq \beta_{NS} - 1$. To see this, recall first that the maximum of (4)

over local models is always reached by some deterministic model, in which a deterministic outcome is assigned to every measurement [and all probabilities in (4) can thus only be equal to 0 or 1]. Hence, deterministic models can only feature $\beta_L \in \mathbb{Z}$. Therefore, it suffices to show that the maximum of (4) over local models satisfies $\beta_L < \beta_{NS}$. This can be proven by *reductio ad absurdum*. Suppose that a local deterministic model attains the value β_{NS} . The model then specifies the outcomes a and b on both sides for all measurements. Equivalently, it can be understood as a definite assignment to every measurement outcome on Alice's and Bob's sides: $v_A(M_a^x) \in \{1,0\}$ and $v_B(M_b^y) \in \{1,0\}$, with $\sum_a v_A(M_a^x) = 1 = \sum_b v_B(M_b^y)$, for all x and y , respectively. If (4) reaches its maximum algebraic value, the assignment map is subject to the constraints $v_A(M_a^x) = v_B(M_1^y) = v_A(M_{a'}^{x'})$ for all (a,x) and $(a',x') \in D_y$. Now, since $\{M_a^x\}$ is in one-to-one correspondence with the projectors $\{\Pi_i^z\}$, v_A can then be thought of as a valid noncontextual deterministic assignment map for $\{\Pi_i^z\}$. This, however, is prohibited because $\{\Pi_i^z\}$ is a KS proof. Thus, one concludes that $\beta_L \leq \beta_{NS} - 1$.

The desired Bell inequality is then

$$\beta(P(a,b|x,y)) \leq \beta_{NS} - 1, \quad (6)$$

with $\beta(P(a,b|x,y))$ defined by (4) and β_{NS} defined by (5). This implies that the quantum correlations obtained above from $|\psi_d\rangle$ feature $p_L = 0$, as they achieve the nonsignalling value of a Bell inequality, which is in turn equal to its algebraic value.

Before concluding this section, we would like to emphasize that this recipe can lead to other, possibly nonequivalent, Bell inequalities. For instance, it is possible to keep Alice's measurements equal to those in the KS proof and replicate them on Bob's side, i.e., $\{M_b^y = (M_a^x)^*$, with $y = x$ and $b = a\}$. Note that then all the projectors needed to enforce the KS constraints on Alice's side by means of perfect correlations appear on Bob's side. Other examples are provided by some proofs that possess inherent symmetries, allowing for peculiar distributions of the contexts in the proof between Alice's and Bob's sides, as is discussed in the next section.

III. A SIMPLE BELL INEQUALITY

The previous recipe is fully general. In this section, in contrast, we apply the ideas just presented to derive a specific Bell inequality maximally violated by quantum mechanics from one of the most elegant KS proofs introduced by Peres and Mermin [13,14]. Apart from being one of the simplest Bell inequalities having this property, its derivation shows how symmetries in the KS proof can be exploited to simplify the previous construction.

The Peres-Mermin (PM) KS proof is based on the set of observables of Table I, also known as the PM square, which can take two possible values, ± 1 . This proof in terms of observables can be mapped into a proof in terms of 24 rank-1 projectors [14,15]. To these projectors we could then apply the formalism of the previous section and derive Bell inequalities maximally violated by quantum correlations of the sort of (6). However, some special features of this particular KS proof allow one to simplify the process and derive a simpler inequality straight from the observables. The key

TABLE I. The Peres-Mermin square. One of the simplest KS proofs was derived by Peres and Mermin [13,14] and is based on the nine observables of this table. The observables are grouped into six groups of three, arranged along columns and rows. X_n , Y_n , and Z_n refer to Pauli matrices acting on qubits $n = 1$ and $n = 2$, which span a four-dimensional Hilbert space. Each group constitutes a complete set of mutually commuting (and therefore compatible) observables, defining thus a context. In this way, there are six contexts, and every observable belongs to two different ones. The product of all three observables in each context is equal to the identity $\mathbb{1}$, except for those of the third row, whose product gives $-\mathbb{1}$. It is impossible to assign numerical values 1 or -1 to each one of these nine observables in a way that the values obey the same multiplication rules as the observables. This, in turn, implies that it is impossible to make a noncontextual assignment to the 24 underlying projectors (not shown) in the table (one common eigenbasis per context, with four eigenvectors each).

	$y = 1$	$y = 2$	$y = 3$	
$x = 1$	Z_2	X_1	$X_1 Z_2$	$=\mathbb{1}$
$x = 2$	Z_1	X_2	$Z_1 X_2$	$=\mathbb{1}$
$x = 3$	$Z_1 Z_2$	$X_1 X_2$	$Y_1 Y_2$	$=-\mathbb{1}$
	$=\mathbb{1}$	$=\mathbb{1}$	$=\mathbb{1}$	\prod

point is that in the PM square each operator appears in two different contexts, one being a row and the other a column. This allows one to distribute the contexts between Alice and Bob in such a way that Alice (Bob) performs the measurements corresponding to the rows (columns) (see also [10]). The corresponding Bell scenario, then, is such that Alice and Bob can choose among three different measurements $x, y \in \{1, 2, 3\}$ of four different outcomes, $a, b \in \{1, 2, 3, 4\}$. Consistent with the PM square, we associate in what follows Alice and Bob's observables x and y with the rows and columns of the square, respectively, and divide the four-value outputs into two bits, $a = (a_1, a_2)$ and $b = (b_1, b_2)$, each of which can take the values ± 1 .

Consider first the following quantum realization: Alice and Bob share two two-qubit maximally entangled states $|\psi_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{34}$, which is equivalent to a maximally entangled state of two four-dimensional systems. Alice possesses systems 1 and 3, and Bob possesses systems 2 and 4. Alice can choose among three different measurements that correspond to the three rows appearing in Table I. If Alice chooses input x , the quantum measurement defined by observables placed in row x is performed. Note that the measurement acts on a four-dimensional quantum state; thus there exist four possible outcomes (one for each eigenvector common to all three observables), which in our scenario are decomposed into two dichotomic outputs. We define a_i to be the value of the observable placed in column $y = i$ for $i = 1, 2$. The value of the third observable in the same row is redundant as it can be obtained as a function of the other two. Equivalently, Bob can choose among three measurements that correspond to the three columns appearing in Table I. If Bob chooses input y , outputs b_j are the values of observables placed in column y and row $x = j$ for $y = 1, 2, 3$ and $j = 1, 2$. This realization attains the algebraic maximum $\beta_Q = \beta_{NS} = 9$

of the linear combination

$$\begin{aligned} \beta = & \langle a_1 b_1 | 1, 1 \rangle + \langle a_2 b_1 | 1, 2 \rangle + \langle a_1 b_2 | 2, 1 \rangle \\ & + \langle a_2 b_2 | 2, 2 \rangle + \langle a_1 a_2 b_1 | 1, 3 \rangle + \langle a_1 a_2 b_2 | 2, 3 \rangle \\ & + \langle a_1 b_1 b_2 | 3, 1 \rangle + \langle a_2 b_1 b_2 | 3, 2 \rangle - \langle a_1 a_2 b_1 b_2 | 3, 3 \rangle, \end{aligned} \quad (7)$$

where $\langle f(a_1, a_2, b_1, b_2) | x, y \rangle$ denotes the expectation value of a function f of the output bits for the measurements x and y .

To prove this statement, let us first focus on the term $\langle a_1 b_1 | 1, 1 \rangle$. Bit b_1 is obtained as the outcome of the measurement of the quantum observable $Z_4 \otimes \mathbb{1}_2$. As the measurement is performed on the maximally entangled state, the state on Alice's side is effectively projected after Bob's measurement onto the eigenspace of $Z_3 \otimes \mathbb{1}_1$ with eigenvalue b_1 . Bit a_1 is defined precisely as the outcome of the measurement of the observable $Z_3 \otimes \mathbb{1}_1$; thus $a_1 = b_1$ and $\langle a_1 b_1 | 1, 1 \rangle = 1$. The same argument applies to the first four terms in (7). Consider now the term $\langle a_1 a_2 b_1 | 1, 3 \rangle$. Bit b_1 is the outcome of the measurement of the observable $Z_4 \otimes X_2$. The state after Bob's measurement is effectively projected on Alice's side onto the eigenspace of $Z_3 \otimes X_1$ with eigenvalue b_1 . Bit $a_1 a_2$ is obtained as the measurement output of the observable $Z_3 \otimes X_1$; thus $a_1 a_2 = b_1$ and $\langle a_1 a_2 b_1 | 1, 3 \rangle = 1$. The same argument applies to the four terms involving products of three bits. The last term $\langle a_1 a_2 b_1 b_2 | 3, 3 \rangle$ requires a similar argument. Bit $a_1 a_2$ is obtained as the output of the operator $Y_3 \otimes Y_1$ (note that the product of the observables associated with a_1 and a_2 is $Y_3 \otimes Y_1$; see Table I). Thus the state is effectively projected onto the eigenspace of $Y_4 \otimes Y_2$ with eigenvalue $a_1 a_2$. Bit $b_1 b_2$ is precisely the measurement outcome of $-Y_4 \otimes Y_2$; thus $a_1 a_2 = -b_1 b_2$ and $\langle a_1 a_2 b_1 b_2 | 3, 3 \rangle = -1$.

We move next to the classical domain to show that the maximum value of polynomial (7) attainable by any local model is $\beta_L = 7$, and thus, the inequality

$$\beta \leq 7, \quad (8)$$

with β defined by (7), constitutes a valid Bell inequality, maximally violated by quantum mechanics. Remarkably, this inequality has already appeared in Ref. [7] in the context of all-versus-nothing nonlocality tests. Computing the local bound $\beta_L = 7$ can easily be performed by brute force (that is, by explicitly calculating the value of β_L for all possible assignments). However, it is also possible to derive it using arguments similar to those in the previous section. In the PM square, each of the nine dichotomic observables belongs to two different contexts, one being a row and the other a column, as mentioned. Therefore, nine correlation terms are needed to enforce the KS constraints. As said, the symmetries of the PM square allow one to split the contexts between Alice and Bob, arranging these correlation terms in a distributed manner. Such correlation terms correspond precisely to the nine terms appearing in (7). Again, the existence of a local model saturating all these terms would imply the existence of a noncontextual model for the PM square, which is impossible.

IV. BOUND ON THE LOCAL CONTENT USING OTHER BELL INEQUALITIES

The scope of this section is to show how the previous construction offers important experimental advantages when

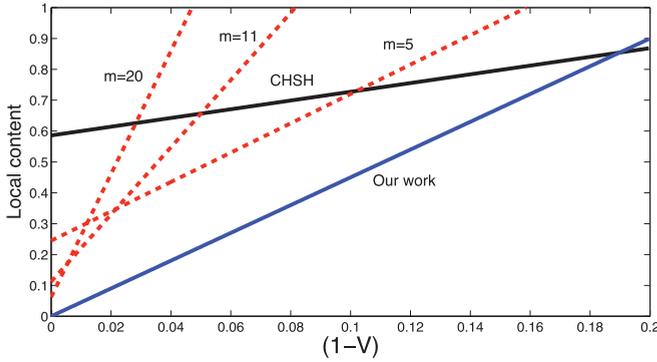


FIG. 3. (Color online) Resistance to noise of different Bell tests. Dashed red curves show the resistance to noise of the chained inequality [20] for different numbers m of measurements. The local content and also the resistance to noise tend to zero when the number of measurements tends to infinity, as expected. Standard Bell inequalities, such as the Clauser-Horne-Shimony-Holt (CHSH) inequality [21], can be violated in a robust manner and with few measurements, but the obtained bound on the local content never goes to zero (in fact, the CHSH inequality is the chained inequality [20] for $m = 2$). Inequality (8) (solid blue curve) in contrast combines all three features: its violation is resistant to noise and requires few measurements, and its bound on the local content is equal to zero in the noise-free case.

deriving bounds on the local content of quantum correlations. First of all, and contrary to some of the examples of quantum correlations with no local part [4], the Bell inequalities derived here not only involve a finite number of measurements but are in addition resistant to noise. Moreover, as shown in what follows, they allow one to obtain experimental bounds on the nonlocal part that are significantly better than those based on other Bell tests.

Let us first consider the Collins-Gisin-Linden-Massar-Popescu inequalities presented in [16]. These inequalities are defined for two measurements of d outcomes. The maximal nonsignalling violation of these inequalities is equal to $\beta_{NS} = 4$, while the local bound is $\beta_L = 2$. The maximal quantum violation of these inequalities is only known for small values

of d [17,18]. A numerical guess for the maximal quantum violation for any d was provided in [19]. This guess reproduces the known values for small d and tends to the nonsignalling value when $d \rightarrow \infty$. Assuming the validity of this guess, a bound on the local content comparable to the experimental value reported in the next section, namely, $p_{L,\max} = 0.218 \pm 0.014$, requires a number of outputs of the order of 200 (see [19]), even in the ideal noise-free situation. Note that the known quantum realization attaining this value involves systems of dimension equal to the number of outputs, that is, 200, and the form of the quantum state is rather complicated. If the quantum state is imposed to be maximally entangled, the maximal quantum violation tends to 2.9681, which provides a bound on the local content of just $p_{L,\max} \approx 0.5195$.

The chained inequalities [4,20], defined in a scenario where Alice and Bob can both perform m measurements of d outcomes, provide a bound on the local content that tends to zero with the number of measurements, $m \rightarrow \infty$ [4]. However, in this limit the nonlocality of the corresponding quantum correlations is not resistant to noise (see Fig. 3), and thus, the use of many measurements requires an almost-noise-free realization. We compare the chained inequalities [20] for $d = 2$ (the simplest case to implement) with our inequality (8) in a realistic noisy situation. The quantum state is written as the mixture of the maximally entangled state, as this state provides the maximal quantum violation of both the chained inequality and inequality (8), with white noise,

$$\rho = V|\psi_d\rangle\langle\psi_d| + (1-V)\frac{\mathbb{1}}{d^2}. \quad (9)$$

The amount of white noise on the state is quantified by $1 - V$. The bound on the local content then reads $p_{L,\max} = \frac{\beta_{NS} - V\beta_Q + (1-V)\beta_{\mathbb{1}}}{\beta_{NS} - \beta_L}$, where $\beta_{\mathbb{1}}$ is the value of the Bell inequality given by white noise with the optimal measurements. We plot the obtained results in Fig. 3. As shown there, the Bell inequality considered here provides better bounds on the local content than the chained inequalities for almost any value of the noise.

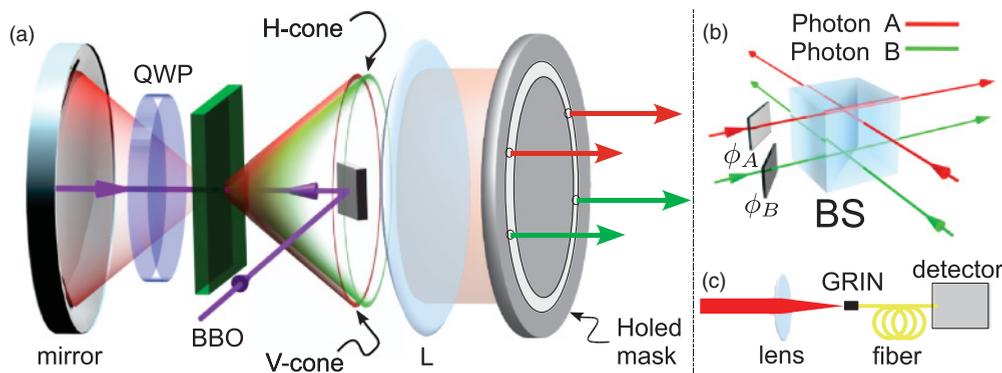


FIG. 4. (Color online) Experimental setup. (a) Source of hyperentangled photon states. The relative phase between states $|HH\rangle_{AB}$ and $|VV\rangle_{AB}$ can be varied by translating the spherical mirror. A lens L located at a focal distance from the crystal transforms the conical emission into a cylindrical one. (b) Scheme for the path measurements. (c) The parametric radiation is coupled into single-mode fibers by a GRIN lens and sent to the detectors.

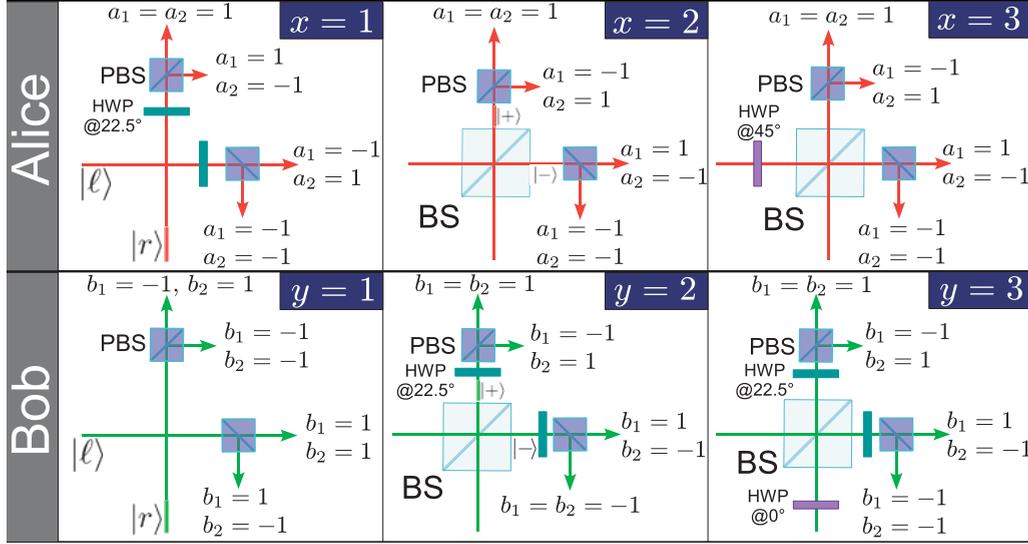


FIG. 5. (Color online) Measurement setups used by Alice and Bob. See text for a detailed explanation of the measurements. BS, beam splitter; PBS, polarizing beam splitter; HWP, half-wave plate.

V. EXPERIMENTAL HIGHLY NONLOCAL QUANTUM CORRELATIONS

We performed a test of inequality (8) with two entangled photons, A and B , generated by spontaneous parametric down conversion (SPDC). We used type-I phase matching with a β -barium-borate (BBO) crystal. The source used a single crystal and a double passage of the UV beam after the reflection on a spherical mirror [see Fig. 4(a)] and generated the hyperentangled state [22]

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle_A|H\rangle_B + |V\rangle_A|V\rangle_B) \otimes \frac{1}{\sqrt{2}}(|r\rangle_A|l\rangle_B + |l\rangle_A|r\rangle_B), \quad (10)$$

where $|H\rangle$ ($|V\rangle$) represents the horizontal (vertical) polarization and $|r\rangle$ and $|l\rangle$ are the two spatial path modes in which each photon can be emitted. Maximally entangled state $|\psi_4\rangle$ between A and B , as defined in Sec. III, is recovered from (10) through the following identification: $|H\rangle_{A,B} \equiv |0\rangle_{1,2}$,

$|V\rangle_{A,B} \equiv |1\rangle_{1,2}$, $|r\rangle_A \equiv |0\rangle_3$, $|l\rangle_A \equiv |1\rangle_3$, $|l\rangle_B \equiv |0\rangle_4$, and $|r\rangle_B \equiv |1\rangle_4$. Therefore, state (10) also allows for the maximal violation of (8).

In the SPDC source, the BBO crystal is shined on by a vertically polarized continuous wave (cw) Ar^+ laser ($\lambda_p = 364$ nm), and the two photons are emitted at degenerate wavelength $\lambda = 728$ nm and with horizontal polarization. Polarization entanglement is generated by the double passage (back and forth, after the reflection on a spherical mirror) of the UV beam. The backward emission generates the so called V cone: the SPDC horizontally polarized photons passing twice through the quarter-wave plate (QWP) are transformed into vertically polarized photons. The forward emission generates the H cone [the QWP behaves almost as a half-wave plate (HWP) for the UV beam]. See Fig. 4(a). Thanks to temporal and spatial superposition, the indistinguishability of the two perpendicularly polarized SPDC cones creates polarization entanglement $(|H\rangle_A|H\rangle_B + |V\rangle_A|V\rangle_B)/\sqrt{2}$. The two polarization entangled photons are emitted over symmetrical directions belonging to the surface of the cone. By selecting two pairs of correlated

TABLE II. Measurement settings. Each row represents a measurement (context). The four states in each row represent the four projectors of each measurement. $a_{1,2}$ and $b_{1,2}$ are the two-bit outcomes of Alice and Bob, respectively. In each state, the first ket refers to polarization, while the second one refers to path. $|\pm\rangle$ correspond to $\frac{1}{\sqrt{2}}(|H\rangle \pm |V\rangle)$ or $\frac{1}{\sqrt{2}}(|r\rangle \pm |l\rangle)$ for polarization or path, respectively.

	Alice			
	$a_1 = -1, a_2 = -1$	$a_1 = -1, a_2 = 1$	$a_1 = 1, a_2 = -1$	$a_1 = 1, a_2 = 1$
$x = 1$	$ -\rangle l\rangle$	$ +\rangle l\rangle$	$ -\rangle r\rangle$	$ +\rangle r\rangle$
$x = 2$	$ V\rangle -\rangle$	$ V\rangle +\rangle$	$ H\rangle -\rangle$	$ H\rangle +\rangle$
$x = 3$	$ H\rangle l\rangle - V\rangle r\rangle$	$ H\rangle l\rangle + V\rangle r\rangle$	$ H\rangle r\rangle - V\rangle l\rangle$	$ H\rangle r\rangle + V\rangle l\rangle$
	Bob			
	$b_1 = -1, b_2 = -1$	$b_1 = -1, b_2 = 1$	$b_1 = 1, b_2 = -1$	$b_1 = 1, b_2 = 1$
$y = 1$	$ V\rangle r\rangle$	$ H\rangle r\rangle$	$ V\rangle l\rangle$	$ H\rangle l\rangle$
$y = 2$	$ -\rangle -\rangle$	$ -\rangle +\rangle$	$ +\rangle -\rangle$	$ +\rangle +\rangle$
$y = 3$	$ +\rangle r\rangle - -\rangle l\rangle$	$ +\rangle r\rangle + -\rangle l\rangle$	$ -\rangle r\rangle - +\rangle l\rangle$	$ -\rangle r\rangle + +\rangle l\rangle$

modes by a four-holed mask [22–24] it is possible to generate path entanglement.

In order to measure the path operators, the four modes of the hyperentangled state are matched on a beam splitter (BS) in a complete indistinguishability condition. This operation corresponds to the projection onto $\frac{1}{\sqrt{2}}(|r\rangle_A + e^{i\phi_A}|l\rangle_A) \otimes \frac{1}{\sqrt{2}}(|r\rangle_B + e^{i\phi_B}|l\rangle_B)$. Suitable tilting of two thin glass plates allows one to set phases ϕ_A and ϕ_B [see Fig. 4(b)]. Photon collection is performed by integrated systems of graded-index (GRIN) lenses and single-mode fibers connected to single-photon counting modules [25,26] [see Fig. 4(c)]. Polarization analysis is performed in each output mode by a polarizing beam splitter (PBS) and a properly oriented HWP. The experimental setup used for each polarization measurement setting is shown in Fig. 5.

The nine terms of Bell polynomial (7) correspond to the different combinations between one of Alice’s three contexts and one of Bob’s three contexts listed in Table II. In the settings $x = 1, 2$ ($y = 1, 2$) Alice (Bob) must project into states that are separable between path and polarization (eigenstates of Pauli operators X and Z). To project into $\{|r\rangle, |l\rangle\}$ the modes are detected without BS. On the other hand, the BS is used to project into $\frac{1}{\sqrt{2}}(|r\rangle \pm |l\rangle)$. PBSs and wave plates have been exploited to project into $\{|H\rangle, |V\rangle\}$ or $\frac{1}{\sqrt{2}}(|H\rangle \pm |V\rangle)$. More details are needed for contexts $x, y = 3$, corresponding to the projection into single-photon Bell states (the two entangled qubits of the Bell state are encoded in polarization and path of the single particle; see Table II). For instance, let us consider the projection on the states $|H\rangle|l\rangle \pm |V\rangle|r\rangle$ and $|V\rangle|l\rangle \pm |H\rangle|r\rangle$ for Alice. By inserting a HWP oriented at 45° on the mode $|l\rangle_A$ before the BS, the previous states become $|V\rangle|\pm\rangle$ and $|H\rangle|\pm\rangle$, respectively. The two BS outputs allow one to discriminate between $|r\rangle + |l\rangle$ and $|r\rangle - |l\rangle$, while the two outputs of the PBSs discriminate $|H\rangle$ and $|V\rangle$.

Table III provides the experimental values of all nine correlations in Bell polynomial (7). The obtained violation for Bell inequality (8) is $\beta_O^{\text{exp}} = 8.564 \pm 0.028$ and provides the upper bound $p_{L\text{max}} = 0.218 \pm 0.014$. At this point it is important to mention that another experimental test of (8) was reported in Ref. [27] in the framework of all-versus-nothing nonlocality tests. The violation in Ref. [27] is compatible (within experimental errors) with the value obtained by our experiment.

TABLE III. Experimental results. Errors were calculated by propagating Poissonian errors of the counts.

Correlation	Experimental result
$\langle a_1 b_1 1, 1 \rangle$	0.9968 ± 0.0032
$\langle a_1 b_2 2, 1 \rangle$	0.9759 ± 0.0058
$\langle a_2 b_1 1, 2 \rangle$	0.9645 ± 0.0068
$\langle a_2 b_2 2, 2 \rangle$	0.941 ± 0.010
$\langle a_1 a_2 b_1 1, 3 \rangle$	0.9705 ± 0.0048
$\langle a_1 a_2 b_2 2, 3 \rangle$	0.9702 ± 0.0049
$\langle a_1 b_1 b_2 3, 1 \rangle$	0.9688 ± 0.0073
$\langle a_2 b_1 b_2 3, 2 \rangle$	0.890 ± 0.013
$\langle a_1 a_2 b_1 b_2 3, 3 \rangle$	-0.888 ± 0.018

TABLE IV. Bounds on the local content of quantum correlations from previous Bell experiments. The selection includes representative experiments testing different forms of nonlocality, or Bell inequalities, in both the bipartite [20,21] and multipartite [28,29] scenarios. Other published experiments, not shown in the table, lead to $p_{L\text{max}} > 0.49$. Note the significant improvement given by the techniques discussed in this work (see also Sec. IV).

Experiment	p_L
Aspect <i>et al.</i> [30]	$\lesssim 0.80$
Wehls <i>et al.</i> [31]	$\lesssim 0.64$
Kiesel <i>et al.</i> [32]	$\lesssim 0.64$
Zhao <i>et al.</i> [33]	$\lesssim 0.60$
Pomarico <i>et al.</i> [34]	$\lesssim 0.49$
This work (and Yang <i>et al.</i> [27])	$\lesssim 0.22$

VI. CONCLUSIONS AND DISCUSSIONS

In this work we have provided a systematic recipe for obtaining bipartite Bell inequalities from every proof of the Kochen-Specker theorem. These inequalities are violated by quantum correlations in an extremal way, thus revealing the fully nonlocal nature of quantum mechanics. We have shown that these inequalities allow establishing experimental bounds on the local content of quantum correlations that are significantly better than those obtained using other constructions. This enabled us to experimentally demonstrate a Bell violation leading to the highly nonlocal bound $p_L \lesssim 0.22$.

The local content p_L of some correlations $P(a, b|x, y)$ can be understood as a measure of their locality, as it measures the fraction of experimental runs admitting a local-hidden-variable description. As mentioned, some of the previously known examples of bipartite inequalities featuring fully nonlocal correlations, i.e., $p_L = 0$, for arbitrary dimensions require an infinite number of measurement settings and are not robust against noise [3,4]. More standard Bell inequalities using a finite number of measurements, such as the well-known Clauser-Horne-Shimony-Holt inequality [21], give a local weight significantly larger than zero even in the noise-free situation. Thus, the corresponding experimental violations, inevitably noisy, have only managed to provide bounds on the local content not smaller than 0.5 (see Table IV). In contrast, the theoretical techniques provided in this work enable the experimental demonstration of highly nonlocal correlations. This explains why the experimental bound provided in this work is significantly better than those of previous Bell tests, even including multipartite ones. In fact, multipartite Greenberger-Horne-Zeilinger tests [11] also in principle yield $p_L = 0$ [4] using a finite number of measurements and featuring robustness against noise. Still, to our knowledge, the reported experimental violations lead to significantly worse bounds on p_L (see Table IV). Our analysis, then, certifies that, in terms of local content, the present bounds allow a higher degree of nonlocal correlations than those reported in [30–34] or in any other previous experiment of our knowledge.

ACKNOWLEDGMENTS

We acknowledge support from Spanish projects FIS2008-05596 and FIS2010-14830, QOIT (Consolider Ingenio 2010), a Juan de la Cierva grant, the European EU FP7 Project

Q-Essence, EU-Project CHIST-ERA QUASAR, EU-Project CHIST-ERA DIQIP, an ERC Starting Grant PERCENT, CatalunyaCaixa, Generalitat de Catalunya, Italian projects

PRIN 2009 of Ministero dell'Istruzione, dell'Universit e della Ricerca and FARI 2010 Sapienza Universit di Roma, and the Wenner-Gren Foundation.

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- [1] J. S. Bell, *Physics* **1**, 195 (1964).
 [2] S. Popescu and D. Rohrlich, *Phys. Lett. A* **166**, 293 (1992).
 [3] A. Elitzur, S. Popescu, and D. Rohrlich, *Phys. Lett. A* **162**, 25 (1992).
 [4] J. Barrett, A. Kent, and S. Pironio, *Phys. Rev. Lett.* **97**, 170409 (2006).
 [5] S. Kochen and E. P. Specker, *J. Math. Mech.* **17**, 59 (1967).
 [6] P. Heywood and M. L. G. Redhead, *Found. Phys.* **13**, 481 (1983); A. Stairs, *Philos. Sci.* **50**, 578 (1983).
 [7] A. Cabello, *Phys. Rev. Lett.* **87**, 010403 (2001).
 [8] G. Brassard, A. Broadbent, and A. Tapp, *Found. Phys.* **35**, 1877 (2005).
 [9] J. H. Conway and S. Kochen, *Found. Phys.* **36**, 1441 (2006).
 [10] K. Horodecki, M. Horodecki, P. Horodecki, R. Horodecki, M. Pawłowski, and M. Bourennane, e-print arXiv:1006.0468.
 [11] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* (Kluwer, Dordrecht, 1989), p. 69.
 [12] A. M. Gleason, *J. Math. Mech.* **6**, 885 (1957).
 [13] N. D. Mermin, *Phys. Rev. Lett.* **65**, 3373 (1990).
 [14] A. Peres, *J. Phys. A* **24**, L175 (1991).
 [15] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, *Phys. Lett. A* **212**, 183 (1996).
 [16] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, *Phys. Rev. Lett.* **88**, 040404 (2002).
 [17] A. Acín, T. Durt, N. Gisin, and J. I. Latorre, *Phys. Rev. A* **65**, 052325 (2002).
 [18] M. Navascués, S. Pironio, and A. Acín, *New J. Phys.* **10**, 073013 (2008).
 [19] S. Zohren and R. D. Gill, *Phys. Rev. Lett.* **100**, 120406 (2008).
 [20] S. L. Braunstein and C. M. Caves, *Ann. Phys. (NY)* **202**, 22 (1990).
 [21] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys. Rev. Lett.* **23**, 880 (1969).
 [22] C. Cinelli, M. Barbieri, R. Perris, P. Mataloni, and F. De Martini, *Phys. Rev. Lett.* **95**, 240405 (2005).
 [23] G. Vallone, E. Pomarico, F. De Martini, and P. Mataloni, *Phys. Rev. Lett.* **100**, 160502 (2008).
 [24] A. Chiuri, G. Vallone, N. Bruno, C. Macchiavello, D. Bruß, and P. Mataloni, *Phys. Rev. Lett.* **105**, 250501 (2010).
 [25] A. Rossi, G. Vallone, A. Chiuri, F. De Martini, and P. Mataloni, *Phys. Rev. Lett.* **102**, 153902 (2009).
 [26] G. Vallone, G. Donati, F. Martini, and P. Mataloni, *Appl. Phys. Lett.* **95**, 181110 (2009).
 [27] T. Yang, Q. Zhang, J. Zhang, J. Yin, Z. Zhao, M. Żukowski, Z.-B. Chen, and J.-W. Pan, *Phys. Rev. Lett.* **95**, 240406 (2005).
 [28] M. Ardehali, *Phys. Rev. A* **46**, 5375 (1992).
 [29] V. Scarani, A. Acín, E. Schenck, and M. Aspelmeyer, *Phys. Rev. A* **71**, 042325 (2005).
 [30] A. Aspect, J. Dalibard, and G. Roger, *Phys. Rev. Lett.* **49**, 1804 (1982).
 [31] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, *Phys. Rev. Lett.* **81**, 5039 (1998).
 [32] N. Kiesel, C. Schmid, U. Weber, G. Tóth, O. Gühne, R. Ursin, and H. Weinfurter, *Phys. Rev. Lett.* **95**, 210502 (2005).
 [33] Z. Zhao, T. Yang, Y.-A. Chen, A.-N. Zhang, M. Żukowski, and J.-W. Pan, *Phys. Rev. Lett.* **91**, 180401 (2003).
 [34] E. Pomarico, J.-D. Bancal, B. Sanguinetti, A. Rochdi, and N. Gisin, e-print arXiv:1101.2313.