

Hybrid ququart-encoded quantum cryptography protected by Kochen-Specker contextuality

Adán Cabello,^{1,2,*} Vincenzo D'Ambrosio,³ Eleonora Nagali,³ and Fabio Sciarrino^{3,4,†}

¹*Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain*

²*Department of Physics, Stockholm University, S-10691 Stockholm, Sweden*

³*Dipartimento di Fisica della "Sapienza" Università di Roma, I-00185 Roma, Italy*

⁴*Istituto Nazionale di Ottica, Consiglio Nazionale delle Ricerche (INO-CNR), I-50125 Florence, Italy*

(Received 22 July 2011; published 12 September 2011)

Quantum cryptographic protocols based on complementarity are not secure against attacks in which complementarity is imitated with classical resources. The Kochen-Specker (KS) theorem provides protection against these attacks, without requiring entanglement or spatially separated composite systems. We analyze the maximum tolerated noise to guarantee the security of a KS-protected cryptographic scheme against these attacks and describe a photonic realization of this scheme using hybrid ququarts defined by the polarization and orbital angular momentum of single photons.

DOI: [10.1103/PhysRevA.84.030302](https://doi.org/10.1103/PhysRevA.84.030302)

PACS number(s): 03.67.Dd, 03.65.Ud, 42.50.Xa, 42.50.Tx

Introduction. Quantum key distribution (QKD) protocols allow two distant parties to share a secret key by exploiting the fundamental laws of quantum mechanics. However, standard quantum cryptographic protocols based on quantum complementarity, such as the Bennett-Brassard 1984 (BB84) protocol [1], are not secure against attacks in which the adversary imitates complementarity with classical resources [2]. Interestingly, BB84-like protocols can be improved to assure “the best possible protection quantum theory can afford” [2] by exploiting the fact that the Bell [3] and Kochen-Specker (KS) [4] theorems show that the outcomes of quantum measurements do not admit local and noncontextual descriptions, respectively. The extra security provided by the Bell theorem has been extensively investigated [5–7]. However, this extra security is based on the assumption that the legitimate parties can perform a loophole-free Bell test, something which is beyond the present technological capabilities and is not expected to be an easy task in the future [8]. A similar problem affects recent proposals combining the KS theorem with entanglement [9,10]. Therefore, it is worth exploring the extra security offered by the KS theorem in situations which require neither entanglement nor composite systems, but only single systems with three or more distinguishable states. For cryptographic purposes, the difference between qubits and systems of higher dimensionality is this: Whereas in qubits different bases are always disjoint, from qutrits onward different bases may share common elements. It is this property which is at the root of the proofs of Bell and KS theorems.

Here we investigate the experimental requirements for obtaining the extra security offered by a KS-protected QKD protocol introduced by Svozil [11], based on the properties of the simplest KS set of states [12]. Hence we propose to implement such a protocol by adopting ququart states encoded in the hybrid polarization-orbital angular momentum four-dimensional space of single photon states [13,14]. For this purpose, we introduce the optical schemes to measure all the states needed to prove KS contextuality. The capability of

encoding a four-dimensional quantum state in a single photon by exploiting these two different degrees of freedom enables us to achieve a high stability and transmission rate in free-space propagation.

Svozil's protocol. The cryptographic protocol introduced by Svozil in [11] is a variation of the BB84 protocol and works as follows: (i) Alice randomly picks a basis from the nine available in Fig. 1 and sends Bob a randomly chosen state of that basis. (ii) Bob, independently from Alice, picks a basis at random from the nine available and measures the system received from Alice. (iii) Bob announces his bases over a public channel, and Alice announces those events in which the state sent belongs to the measured basis. Therefore, the probability of adopting the same basis is $\frac{1}{9}$. (iv) Alice and Bob exchange some of the remaining matching outcomes over a public channel to ensure that nobody has spied their quantum channel. (v) Alice and Bob encode the four outcomes by using four different symbols. As a result, for each successful exchange Bob and Alice share a common random key.

The advantage of this protocol over the BB84 protocol is that it is protected by the KS theorem against attacks in which the adversary replaces the quantum system with a classical one. These attacks can be described using a classical toy model [2,11] in which, in step (i), Alice is actually picking one of nine differently colored eyeglasses (instead of one of the nine different bases in Fig. 1) and picking a ball from an urn (instead of picking one of the 18 states in Fig. 1) with two color symbols in it (corresponding to the two bases the state belongs to). Each one of the nine differently colored eyeglasses allows her to see only one of the nine different colors. To reproduce the quantum predictions: (a) each of the balls must have one symbol $S_i \in \{1, 2, 3, 4\}$ written in two different colors chosen among the 18 possible pairs. Her choice of eyeglass decides which symbols Alice can see. (b) All colors are equally probable and, for a given color, the four symbols are equally probable. In step (ii), Bob is actually picking one of nine differently colored eyeglasses and reading the corresponding symbol. A classical strategy like this one can successfully imitate the quantum part of the BB84 protocol (see [2] for details) but not the protocol described above. The reason is that the requirements (a) and (b) cannot be satisfied simultaneously. Figure 1 shows how to

*adan@us.es

†fabio.sciarrino@uniroma1.it <http://quantumoptics.phys.uniroma1.it>

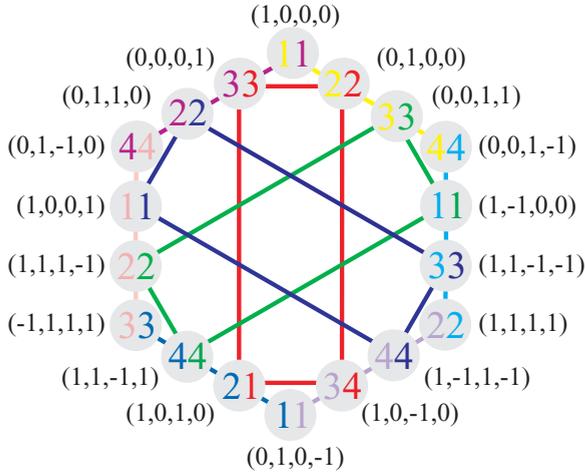


FIG. 1. (Color online) The protocol is based on a KS set of 18 states which can be grouped in 9 bases represented by 9 colors. Every state belongs to two different bases. No set of 18 balls can have all the properties required to imitate the KS set; at least two balls must have different symbols. Therefore, the imitation can be detected.

prepare 18 balls with the minimum number of balls not having the same symbol.

Experimental requirements. As shown in Fig. 1, the minimum number of balls not having the same symbol is two out of 18. A ball attack can be detected only in those runs in which Alice and Bob pick differently colored eyeglasses. Therefore, for the set in Fig. 1, the trace of such an attack will be a $\frac{2}{18}$ probability of Alice picking a symbol such that the corresponding interlinked symbol (seen only with differently colored eyeglasses) is different. As a consequence, to demonstrate that the experimental results cannot actually be imitated with balls and to experimentally certify the extra security of this KS-based QKD protocol, we need an experimental probability w of wrong state identification, defined as the probability that Bob makes a wrong identification of the state sent by Alice when Bob has successfully measured in a correct basis, of $w < \frac{1}{9} \approx 0.111$.

Implementation using polarization- and orbital-angular-momentum-encoded ququarts. Here we propose a scheme for the experimental implementation of the KS-protected QKD protocol. To test its feasibility, we need to prepare the 18 states, measure each of them in two different bases, and obtain an average value of w over the 18×2 possibilities. The condition which must be fulfilled is $w < 0.111$, which corresponds to a mean fidelity value of the transmission of the state of $F = 0.889$. In addition, to check that any intercept and resend strategy causes a disturbance, one should be able to measure what happens when the states are measured in the wrong basis. While in the correct basis the probabilities for the four possible outcomes are (in the ideal case) 0, 0, 0, and 1, in the wrong basis they are either 0, 0, $\frac{1}{2}$, and $\frac{1}{2}$ or $0, \frac{1}{4}, \frac{1}{4},$ and $\frac{1}{2}$.

Svozil's protocol uses nine sets of four-dimensional states defining a 18-state KS set. We propose encoding four-dimensional quantum states by exploiting two different degrees of freedom of the same particle, an approach that allows us to achieve higher efficiency in the transmission process. It has recently been demonstrated that ququart states can be efficiently generated by manipulating the polarization and

orbital angular momentum (OAM) of a single photon [13]. In particular, we consider a bidimensional subset of the infinite-dimensional OAM space, denoted as o_1 , spanned by states with OAM eigenvalue $m = \pm 1$ in units of \hbar . According to the nomenclature $|\varphi, \phi\rangle = |\varphi\rangle_\pi |\phi\rangle_{o_1}$, where $|\cdot\rangle_\pi$ and $|\cdot\rangle_{o_1}$ stand for the photon quantum state “kets” in the polarization and OAM degrees of freedom, the logic ququart basis can be rewritten as

$$\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\} \rightarrow \{|H, +1\rangle, |H, -1\rangle, |V, +1\rangle, |V, -1\rangle\}, \quad (1)$$

where H (V) refers to horizontal (vertical) polarization. Following the same convention, the OAM equivalent of the basis $|H\rangle$ and $|V\rangle$ is then defined as $|h\rangle = \frac{1}{\sqrt{2}}(|+1\rangle + |-1\rangle)$ and $|v\rangle = \frac{i}{\sqrt{2}}(|+1\rangle - |-1\rangle)$. Finally, the $\pm 45^\circ$ angle “antidiagonal” and “diagonal” linear polarizations are hereafter denoted by the kets $|A\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $|D\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$, while the OAM equivalent is denoted by $|a\rangle = (|h\rangle + |v\rangle)/\sqrt{2}$ and $|d\rangle = (|h\rangle - |v\rangle)/\sqrt{2}$. It is convenient to work with Laguerre-Gauss laser modes ($LG_{0,\pm 1}$) as OAM eigenstates since, in this case, the states $(|h\rangle, |v\rangle, |a\rangle, |d\rangle)$ will result as the Hermite-Gauss modes ($HG_{1,0}, HG_{0,1}$) along the axes and rotated by 45° . This feature allows us to easily transform the states by an astigmatic laser mode converter [15]. We stress that by choosing a bidimensional subspace of OAM we avoid detrimental effects on the state due to the radial contribution in the free propagation and Gouy-phases associated with different OAM values [16]. Hence, a hybrid approach for the encoding of a ququart state, based on OAM and polarization, leads to a higher stability for the single photon propagation compared to a qudit implemented only by adopting the OAM degree of freedom. According to the previous definitions, a state (a_1, a_2, a_3, a_4) of the KS set is implemented as

$$a_1|H, +1\rangle + a_2|H, -1\rangle + a_3|V, +1\rangle + a_4|V, -1\rangle. \quad (2)$$

The coefficients a_i for each state are shown in Table I, along with the settings needed to analyze each basis.

Generation. Figure 2 shows the optical schemes for the generation and detection of any ququart state of the KS set. The generation of the states can be achieved by adopting a spontaneous parametric down conversion (SPDC) source of pair of photons, as in Fig. 2(a), where we consider a collinear generation of couples $|H\rangle|V\rangle$, where one of the two photons acts as a trigger for the heralded generation of a single photon to be sent to the experimental setup. As in [13], the manipulation of the OAM degree of freedom can be achieved by adopting the q -plate device [16,17]. On the polarization, the q plate acts as a half-wave plate, while on the OAM it imposes a shift on the eigenvalue $m = \pm 2q$, where q is an integer or half-integer number determined by the (fixed) pattern of the optical axis of the device. In order to manipulate the OAM subspace $o_1 = \{|+1\rangle, |-1\rangle\}$, a q plate with topological charge $q = 1/2$ should be adopted [18]. Interestingly, the fact that the q plate can entangle or disentangle the OAM and polarization degrees of freedom can be exploited for the preparation of any ququart states. In order to generate all the states of the KS set, it is sufficient to exploit a technique based on a quantum transducer

TABLE I. The ququart states that compose the KS set are divided in nine basis and encoded in polarization and orbital angular momentum degrees of freedom by adopting the devices in Fig. 2. Column E identifies whether or not the states are entangled states of the two degrees of freedom. Column S specifies the experimental setup to be adopted for the analysis. Column MC gives the angle between the horizontal axis and the orientation of the cylindrical lenses of the mode converter MC. Column QWP indicates the angle of the quarter wave plate to be inserted after the PSI in Fig. 2(b) for the analysis of bases IV and IX. WP refers to the type of wave plate to be inserted in the setup: H means half-wave plate and Q means quarter-wave plate.

Set	Logic	E	S	PSI1	PSI2	QWP	MC	WP
I	(1,0,0,0),(0,1,0,0)		b				$\frac{\pi}{4}$	
	(0,0,1,1),(0,0,1,-1)		b				$\frac{\pi}{4}$	
II	(1,1,1,1),(1,1,-1,-1)		c				$\frac{\pi}{4}$	H $\frac{\pi}{8}$
	(1,-1,0,0),(0,0,1,-1)		c				$\frac{\pi}{4}$	H $\frac{\pi}{8}$
III	(1,1,1,1),(1,-1,1,-1)		b				$\frac{\pi}{4}$	
	(1,0,-1,0),(0,1,0,-1)		b				$\frac{\pi}{4}$	
IV	(-1,1,1,1),(1,1,-1,1)	✓	b	✓		$\frac{\pi}{4}$	0	
	(1,0,1,0),(0,1,0,-1)		b	✓		$\frac{\pi}{4}$	0	
V	(1,0,0,1),(0,1,-1,0)	✓	c	✓	✓		$\frac{\pi}{4}$	H $\frac{\pi}{8}$
	(1,1,1,-1),(-1,1,1,1)	✓	c	✓	✓		$\frac{\pi}{4}$	H $\frac{\pi}{8}$
VI	(1,0,0,1),(0,1,1,0)	✓	c		✓		0	Q $\frac{\pi}{4}$
	(1,1,-1,-1),(1,-1,1,-1)		c		✓		0	Q $\frac{\pi}{4}$
VII	(1,1,1,-1),(1,1,-1,1)	✓	c		✓		0	Q $\frac{\pi}{4}$
	(0,0,1,1),(1,-1,0,0)		c		✓		0	Q $\frac{\pi}{4}$
VIII	(0,0,0,1),(1,0,1,0)		c					H $\frac{\pi}{8}$
	(1,0,-1,0),(0,1,0,0)		c					H $\frac{\pi}{8}$
IX	(0,1,-1,0),(0,1,1,0)	✓	b	✓		$\frac{\pi}{4}$	0	
	(1,0,0,0),(0,0,0,1)		b	✓		$\frac{\pi}{4}$	0	

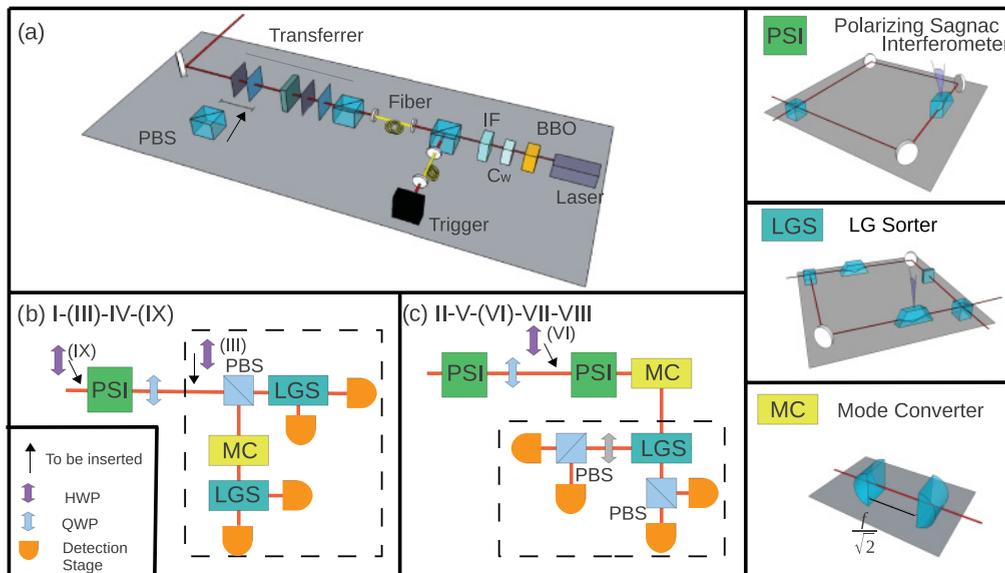


FIG. 2. (Color online) (a) Setup for the generation of ququart states: One of the two photons emitted by SPDC acts as a trigger, while the other one is sent to a polarizing beam splitter (PBS), wave plates, and a quantum transferrer based on the q plate in order to generate the desired ququart. (b) Setup for the analysis of bases (I-III-IV-IX): The setup in the dotted rectangle analyzes the four states of basis I; basis III can be measured by inserting a half-wave plate (HWP) at $\pi/8$ before the PBS. A polarizing Sagnac interferometer (PSI) and a quarter-wave plate are needed to analyze bases IV and IX (adding a HWP at $\pi/8$ before the PSI). (c) Setup for the analysis of bases (II-V-VI-VII-VIII): The part in the dotted rectangle is suitable to sort the four states of all the bases (the gray wave plate can be a HWP or a QWP depending on the particular basis as shown in Table I); this part is sufficient to analyze basis VIII. Basis II can be analyzed by adding a mode converter (MC). Using a PSI before the MC makes it possible to analyze bases VI (adding a HWP at $\pi/8$) and VII. Finally, the states of basis V can be sorted by an additional PSI and QWP. The pictures in the three boxes on the right represent the Sagnac interferometer, the LG mode sorter, and the cylindrical lens mode converter, respectively. The detection stage consists of a q plate, a single-mode fiber, and a detector.

$\pi \rightarrow o_1$ described in [13]. The OAM eigenmodes produced in this way are not exactly LG modes but hypergeometric Gaussian ones [19]. Since some of the detection schemes are based on the properties of Laguerre-Gaussian modes, this fact will lead, in some cases, to a detection efficiency of around 80%. Thus, in order to avoid noise due to different OAM order contributions, it is sufficient to insert in the detection stage a q plate and a single-mode fiber connected to the detector (see Fig. 2).

Measurement of the KS bases. The bases involved in the KS set have different structures as shown in Table I. They can be classified in three groups, depending on whether they are composed of separable, entangled (between polarization and OAM), or both separable and entangled states.

The detection setup is shown in Figs. 2(b) and (c). Their components are a polarizing Sagnac interferometer with a Dove prism (PSI) [20], an astigmatic laser mode converter (MC) [15], and a Laguerre-Gauss mode sorter (LGS) [21]. The PSI consists of a Sagnac interferometer with a polarizing beam splitter as input-output gate and a Dove prism that intercepts the two counterpropagating beams and can be rotated around the optical axes. This scheme allows us, under appropriate conditions, to transform an entangled state into a separable one. In this case, the prism must be rotated in order to add a phase shift of $\Delta\phi = \pi/2$ between $|H\rangle$ and $|V\rangle$ ($\alpha = \pi/8$ in Fig. 2). For example, the states of basis IV are transformed into $(|L, a\rangle, |L, d\rangle, |R, +1\rangle, |R, -1\rangle)$. The MC consists of two cylindrical lenses (with the same focal length f) at distance $f/\sqrt{2}$. It allows us to convert the HG states $(|a\rangle, |d\rangle)$ into

$(|+1\rangle, |-1\rangle)$ and, if rotated by 45° along the optical axes, to convert $(|h\rangle, |v\rangle)$ into $(|+1\rangle, |-1\rangle)$ [15]. The LGS consists of a Mach-Zehnder interferometer with a Dove prism in each arm. The two prisms are rotated by $\beta = \pi/4$ with respect to each other. A phase plate ($\psi = \pi/2$) in one of the two arms allows us to send $|+1\rangle$ and $|-1\rangle$ in the two different output ports of the Mach-Zehnder. States belonging to sets I-III-IV-IX can be analyzed by adopting the scheme reported in Fig. 2(b) with some slight modifications related to the specific basis to be measured. The scheme in Fig. 2(c) leads to the analysis of bases II-V-VI-VII-VIII. All the details on the settings of the different measurement devices are in Table I.

Conclusions. Device-independent QKD based on loophole-free Bell tests are still far in the future. It is therefore worth investigating whether quantum contextuality can produce some extra protection to BB84-like protocols which do not use entangled states. Here we have presented a proposal to demonstrate a quantum contextuality-based extra protection against a particular attack, requiring neither composite systems nor entangled states.

Acknowledgments. We acknowledge helpful discussions with E. Karimi and L. Marrucci. This work was supported by the MICINN Project No. FIS2005-07689, the Wenner-Gren Foundation, FIRB Futuro in Ricerca-HYTEQ, and Project PHORBITECH of the Future and Emerging Technologies (FET) program within the Seventh Framework Programme for Research of the European Commission, under FET-Open Grant No. 255914.

-
- [1] C. H. Bennett and G. Brassard, in *Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India* (IEEE, New York, 1984), p. 175.
- [2] K. Svozil, *Am. J. Phys.* **74**, 800 (2006).
- [3] J. S. Bell, *Physics* **1**, 195 (1964).
- [4] S. Kochen and E. P. Specker, *J. Math. Mech.* **17**, 59 (1967).
- [5] A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
- [6] J. Barrett, L. Hardy, and A. Kent, *Phys. Rev. Lett.* **95**, 010503 (2005).
- [7] A. Acín *et al.*, *Phys. Rev. Lett.* **98**, 230501 (2007).
- [8] W. Rosenfeld *et al.*, *Adv. Sci. Lett.* **2**, 469 (2009).
- [9] A. Cabello, *Phys. Rev. Lett.* **104**, 220401 (2010).
- [10] K. Horodecki *et al.*, e-print [arXiv:1006.0468](https://arxiv.org/abs/1006.0468).
- [11] K. Svozil, in *Physics and Computation 2010*, edited by H. Guerra (University of Azores, Portugal, Ponta Delgada, 2010), p. 235.
- [12] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, *Phys. Lett. A* **212**, 183 (1996).
- [13] E. Nagali, L. Sansoni, L. Marrucci, E. Santamato, and F. Sciarrino, *Phys. Rev. A* **81**, 052317 (2010).
- [14] E. Nagali *et al.*, *Phys. Rev. Lett.* **105**, 073602 (2010).
- [15] M. W. Beijersbergen, L. Allen, H. E. L. O. van der Veen, and J. P. Woerdman, *Opt. Commun.* **96**, 123 (1993).
- [16] E. Nagali *et al.*, *Opt. Express* **17**, 18745 (2009).
- [17] E. Nagali *et al.*, *Phys. Rev. Lett.* **103**, 013601 (2009).
- [18] S. Slussarenko *et al.*, *Opt. Express* **19**, 4085 (2011).
- [19] E. Karimi *et al.*, *Opt. Lett.* **21**, 3053 (2007).
- [20] S. Slussarenko, V. D'Ambrosio, B. Piccirillo, L. Marrucci, and E. Santamato, *Opt. Express* **18**, 27205 (2010).
- [21] H. Wei *et al.*, *Opt. Commun.* **223**, 117 (2003).