Hyperentanglement witness

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A new criterion to detect the entanglement present in a *hyperentangled state*, based on the evaluation of an entanglement witness, is presented. We show how some witnesses recently introduced for graph states, measured by only two local settings, can be used in this case. We also define a new witness W_3 that improves the resistance to noise by increasing the number of local measurements.

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I. INTRODUCTION

Quantum entanglement represents the basic property underlying many quantum computation processes and quantum cryptographic schemes. It guarantees in principle secure cryptographic communications and a huge speedup of some important computation tasks. In this respect entanglement represents the basis of the exponential parallelism of the future quantum computers. By using optical techniques entanglement has been realized in many experiments, either with quantum states based on two [1], four [2–4], or even six photons [5], or, more recently, with multiphoton states, containing more than 10 000 entangled particles [6].

By hyperentanglement more degrees of freedom (DOFs) of the photons are involved and entangled states spanning a high-dimensional Hilbert space can be created [7–10]. A hyperentangled (HE) state encoded in n DOF's is expressed by the product of n Bell states, one for each DOF. Double Bell HE states of two photons (i.e., with $n=2$) are currently realized in laboratories and able to perform tasks that are usually not achievable with normally entangled states. Among many applications, the realization of a complete Bell state analysis [11–13], and the recently realized enhanced dense coding [14], are particularly worthy of note. By operating with HE states of two photons and n independent DOFs, we are able to encode the information in $2n$ qubits. This significantly reduces the typical decoherence problems of multiphoton states based on the same number of qubits and dramatically increases the detection efficiency. HE states of increasing size are also important for the realization of advanced quantum nonlocality tests and represent a viable resource to increase the power of computation of a scalable quantum computer operating in the one-way model [15,16]. Indeed, it has been recently demonstrated that efficient four-qubit two-photon cluster states are easily created starting from two-photon HE states [17–20].

The analysis of multiqubit entangled states performed by quantum state tomography is particularly demanding since the number of required measurements scales exponentially with the number of qubits. Furthermore, in practical realizations, entanglement is degraded by decoherence and by any dissipation processes deriving from the unavoidable coupling with the environment. Since entanglement is an expensive resource, its efficient detection with the minimum number of measurements is a crucial issue, and new efficient analysis tools are necessary to characterize the entanglement of a particular multipartite state. The method of the "entanglement witness" [21,22] (see Fig. 1 for a geometrical representation of an entanglement witness), first demonstrated for entangled states of two photons [23], allows us to assess the presence of entanglement by using only a few local measurements. After its introduction the use of entanglement witness was extended to the detection of entanglement of various kinds of four-qubit entangled states [3,4,17,24] and N-qubit cluster states. At the same time much effort was spent in the study of the entanglement witness operator properties [25–29] and their nonlinear generalization [30,31].

In the present paper we address the timely question of finding a criterion to determine the presence of entanglement in HE states and introduce an entanglement witness which enables us to detect entanglement in these states. This criterion is different from those used in the case of multiparticle

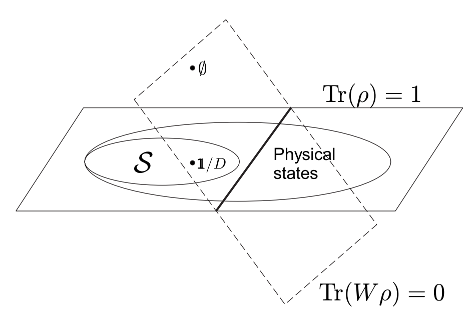

FIG. 1. Geometrical representation of witnesses in the Hilbert space of Hermitian operators ρ. In this space the scalar product is defined by $\langle\rho,\sigma\rangle\equiv\mathrm{Tr}(\rho^\dagger\sigma)$. Physical states (i.e., unit norm positive Hermitian operators) lie on the hyperplane defined by $\langle\mathbb{1},\rho\rangle=\mathrm{Tr}(\rho)=1$. The convex set $\mathcal{S}$ represents the separable states. We show also the completely mixed state $\mathbb{1}/D\in\mathcal{S}$. The (Hermitian) witness W identifies a hyperplane (passing through the null state $\emptyset$) by the equation $\langle W,\rho\rangle\equiv\mathrm{Tr}(W\rho)=0$ and splits the whole Hilbert space into two subsets, one of them containing $\mathcal{S}$. All the separable states $\sigma\in\mathcal{S}$ thus satisfy the condition $\langle\sigma,W\rangle\geqslant 0$ and lie on one side of the hyperplane. The physical states satisfying the condition $\langle\rho,W\rangle<0$ lie on the other side and thus are entangled.

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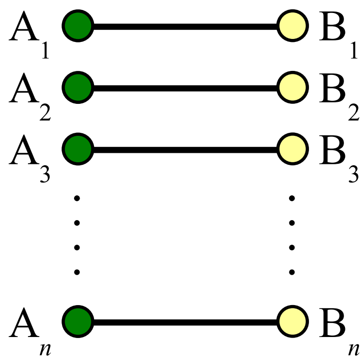

FIG. 2. (Color online) A hyperentangled state represented as a graph state where a dot corresponds to a qubit and the links represent the entanglement existing between the corresponding qubits. Dark (green) and bright (yellow) dots represent the degrees of freedom of particles A and B, respectively.

entangled states, where each qubit is encoded in a different particle. The difference resides in the different partitions that can be made in the two cases, as will be shown in the following section.

II. WITNESS FOR HYPERENTANGLED STATE

Let us consider the generic DOF A_j (B_j), with $j=1,\ldots,n$, of particle A (B). Each DOF spans a two-dimensional Hilbert space (i.e., it is equivalent to a qubit) whose basis is $\{|0\rangle_{A_j},|1\rangle_{A_j}\}$ ($\{|0\rangle_{B_j},|1\rangle_{B_j}\}$) for particle A (B). In this way each particle encodes exactly n qubits. Let us define

$$\mathcal{U}\equiv\{A_1,B_1,\ldots,A_n,B_n\}, \tag{1}$$

the set containing the entire number of DOFs. The (pure) HE state is written as

$$|\Xi\rangle=|\phi^+\rangle_{A_1B_1}|\phi^+\rangle_{A_2B_2}\cdots|\phi^+\rangle_{A_nB_n}, \tag{2}$$

where

$$|\phi^+\rangle_{A_jB_j}\equiv\frac{1}{\sqrt{2}}(|0\rangle_{A_j}|0\rangle_{B_j}+|1\rangle_{A_j}|1\rangle_{B_j}) \tag{3}$$

represents a maximally entangled Bell state. In general, $|\phi^+\rangle$ can be replaced with any maximally entangled state (this corresponds to applying single-qubit unitaries on the HE state). In the language of graph states, a HE state can be interpreted as a graph state up to a Hadamard gate applied to each qubit A_j (see Fig. 2). Let us define now the following $N=2n$ operators:

$$S_{2j}=Z_j^AZ_j^B, \quad S_{2j-1}=X_j^AX_j^B, \quad j=1,\ldots,n, \tag{4}$$

where Z_j^A,X_j^A (Z_j^B,X_j^B) are, respectively, the Pauli matrices σ_z and σ_x acting on the jth DOF of particle A (B). By using the stabilizer formalism [32] the HE state can also be defined as the state satisfying

$$S_k|\Xi\rangle=|\Xi\rangle, \quad \forall\ k=1,\ldots,N. \tag{5}$$

In general we can define the *stabilizer* basis as

$$S_k|\mathbf{s}\rangle=(-1)^{s_k}|\mathbf{s}\rangle, \quad \forall\ k=1,\ldots,N,$$

$$|\mathbf{s}\rangle\equiv|[s_1,s_2,\ldots,s_N]\rangle, \quad s_k=0,1, \tag{6}$$

and $|\Xi\rangle\equiv|[0,0,\ldots,0]\rangle$.

How can we detect entanglement in this case? And also, which kind of entanglement would we like to detect? The entanglement witness method is based on the introduction of a witness W, i.e., a Hermitian operator whose expectation value is non-negative for a generic separable state, while it is negative for the entangled state we want to detect. Since the witness is defined up to a multiplicative positive constant, here and in the following we fix the normalization of a generic witness W for the HE state by requiring that

$$\langle\Xi|W|\Xi\rangle=-1. \tag{7}$$

The advantages of this choice will become evident when the resistance to noise of the entanglement witness is evaluated (see Sec. III).

Let us define the entanglement we want to discriminate. A state $|\varphi\rangle$ is separable (in the hyperentangled sense) if it satisfies the following condition:

$$\exists\ j \text{ such that } |\varphi\rangle=|\varphi_1\rangle_{A_j\mathcal{I}}|\varphi_2\rangle_{B_j\mathcal{J}}. \tag{8}$$

In this equation $\{\mathcal{I},\mathcal{J}\}$ represents a generic bipartition of the set $\mathcal{T}_j\equiv\{A_1,B_1,\ldots,A_n,B_n\}\backslash\{A_j,B_j\}$, so that $\mathcal{I}\cup\mathcal{J}=\mathcal{T}_j$ and $\mathcal{I}\cap\mathcal{J}=\emptyset$.

Definition. A (mixed) state is defined to be *hyperentangled* in n degrees of freedom if it is separately entangled in each of them and cannot be written as a mixture of states that satisfy (8).

In this way the possibility that a classical mixture of two or more states that are not entangled in each DOF, such as those satisfying Eq. (8) can be interpreted as hyperentangled, is avoided. This definition of separability is different from the usual definition used in multiparticle entanglement, where separability is referred to every possible partition of the qubits. In the hyperentanglement case a state is separable if it can be expressed by a partition that separates the same DOF, as written in Eq. (8). This condition is weaker than the usual multiparticle entanglement as can be immediately seen by looking at the state $|\Xi\rangle$ in Eq. (2): this is a hyperentangled state if $A_1,\ldots A_n$ ($B_1,\ldots,B_n$) refer to the different DOFs of particle A (B). However, if any A_j (or B_j) represents a different particle, the state $|\Xi\rangle$ is no longer completely entangled (or "genuinely multiparticle entangled").

The first condition is the entanglement in every DOF. We can then measure n entanglement witnesses. For each DOF j we can introduce a witness given by[1]

$$\mathcal{W}^{(j)}=\mathbb{1}_2-S_{2j}-S_{2j-1}. \tag{9}$$

If all of them are negative,

[1]The choice of this witness at this stage is arbitrary. Other witnesses, such as $\mathcal{W}^{(j)}=\mathbb{1}_2-2|\phi^+\rangle_{A_jB_j}\langle\phi^+|=(1/2)(\mathbb{1}_2-S_{2j}-S_{2j-1}-S_{2j}S_{2j-1})$ can be chosen. The witness chosen in Eq. (9) is useful in terms of the measurement settings, as shown in the following.

$$\mathrm{Tr}(\mathcal{W}^{(j)}\rho) < 0, \quad \forall \ j = 1, \ldots, n, \tag{10}$$

we know that there is entanglement in each DOF. Hence all the reduced matrices

$$\rho_j = \mathrm{Tr}_{\mathcal{U}\backslash A_j B_j}(|\Xi\rangle\langle\Xi|) \tag{11}$$

obtained by tracing all the DOFs but A_j and B_j are entangled.

In the case of pure states the previous condition assures that each DOF of A is entangled with the corresponding one of B. However, this condition, although necessary, is not sufficient to demonstrate that the state cannot be created by a classical (i.e., mixed) superposition of states that are not entangled in all the DOFs. This is clearly explained by a simple example. Let us consider the case with $n=2$ and two states

$$|\psi_1\rangle = |0\rangle_{A_1}|0\rangle_{B_1}|\phi^+\rangle_{A_2B_2},$$

$$|\psi_2\rangle = |\phi^+\rangle_{A_1B_1}|0\rangle_{A_2}|0\rangle_{B_2}. \tag{12}$$

They are not entangled in every DOF since $|\psi_1\rangle$ ($|\psi_2\rangle$) is separable in the first (second) DOF. Then the two states are not hyperentangled:

$$\langle\psi_1|\mathcal{W}^{(2)}|\psi_1\rangle = \langle\psi_2|\mathcal{W}^{(1)}|\psi_2\rangle = -1,$$

$$\langle\psi_1|\mathcal{W}^{(1)}|\psi_1\rangle = \langle\psi_2|\mathcal{W}^{(2)}|\psi_2\rangle = 0. \tag{13}$$

However, by taking the mixture of these two states with equal weights, $\rho' = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$, we obtain

$$\mathrm{Tr}(\mathcal{W}^{(1)}\rho') = \mathrm{Tr}(\mathcal{W}^{(2)}\rho') = -\frac{1}{2}. \tag{14}$$

In this case the mixture of two non-HE states is entangled in every DOF. Note that this feature does not depend on the particular choice of the witness $\mathcal{W}^{(j)}$ in Eq. (10).

The correct identification of a hyperentangled state can be obtained by introducing a *hyperentanglement witness* for the HE state $|\Xi\rangle$:

$$\widetilde{W} = \mathbb{1} - 2|\Xi\rangle\langle\Xi| = \mathbb{1} - \frac{2}{2^N}\sum_{\{s_k\}} S_1^{s_1}S_2^{s_2}\cdots S_N^{s_N}. \tag{15}$$

In the Appendix we will show that

$$|\langle\Xi|\varphi\rangle|^2 \leqslant \frac{1}{2} \tag{16}$$

for all the states $|\varphi\rangle$ that satisfy Eq. (8). The expectation value of $\widetilde{W}$ is thus negative for $|\Xi\rangle$ but is positive for all the states expressed in the form (8), and thus it is positive for all their mixtures. For example, it can be easily verified that for the state ρ' in (14) it holds that $\mathrm{Tr}(\widetilde{W}\rho')=0$ and hence ρ' is (correctly) not hyperentangled.

Given a witness $\widetilde{W}$, other witnesses W' can be derived on the basis of the following argument [33,34]. If we can find a constant $\alpha > 0$ such that the operator $\mathcal{O} \equiv W' - \alpha\widetilde{W}$ is positive definte [i.e., $\mathrm{Tr}(\mathcal{O}\rho) \geqslant 0$, $\forall\ \rho$], then W' is a witness. In fact, if $\mathrm{Tr}(W'\rho) < 0$ on a generic state ρ it turns out that $\mathrm{Tr}(\widetilde{W}\rho) \leqslant (\frac{1}{\alpha})\mathrm{Tr}(W'\rho) < 0$ and thus ρ is entangled.

Following this observation, it is shown in [33,34] that the operators

$$W_1 = (N-1) - \sum_{k=1}^{N} S_k, \tag{17}$$

$$W_2 = 3 - 2\left(\prod_{\text{odd } k}\frac{S_k+1}{2} + \prod_{\text{even } k}\frac{S_k+1}{2}\right) \tag{18}$$

are witnesses for a generic connected graph state. The same argument can be repeated here, since it uses only the stabilizer equation. In fact, even in the case of a HE state (or whatever state is defined in terms of the stabilized equation) the operators $W_1 - \widetilde{W}$ and $W_2 - \widetilde{W}$ are positive definite. This is simply checked in the *stabilizer* basis $|\mathbf{s}\rangle$, where they are diagonal and their eigenvalues are non-negative.

We can also introduce here another witness by using the same argument:

$$W_3 = 2 - 3\prod_{j=1}^{n}\left(\frac{1 + S_{2j-1} + S_{2j}}{3}\right). \tag{19}$$

In order to demonstrate that W_3 is a witness, let us consider $W_3' = c_0 - 3\Pi_{j=1}^{n}[(1+S_{2j-1}+S_{2j})/3]$ and calculate the lowest eigenvalues of $\mathcal{O}_3 \equiv W_3' - \alpha\widetilde{W}$. If they are positive then $\mathcal{O}_3$ is positive definite. The lowest eigenvalues are $\lambda_1 = c_0 - \alpha - 3 + 2\alpha$ for $|\Xi\rangle$ and $\lambda_2 = c_0 - \alpha - 1$ for a state with only one s_k equal to 1. They are equal when $\alpha = 1$ and are both positive when $c_0 \geqslant 2$.

The witness (19) is built by considering all the possible products of stabilizers where, for each DOF, we can measure, $\mathbb{1}$, XX, or ZZ. The four witnesses above differ from each other with respect to the number of measurement settings, i.e., their *local decompositions* [34] are different. We remember that the local decomposition of W is defined by the equation $W = \Sigma_i \mathcal{M}_i$, where $\mathcal{M}_i$ is measured by a different local measurement setting $\{O^{(k)}\}_{k=1}^{N}$. Each $\mathcal{M}_i$ then consists of the simultaneous measurements of $O^{(k)}$ on the corresponding qubit k.

In the case of W_1 and W_2 the local decomposition consists of two terms: the first is computed by the local setting $X_1X_2X_3X_4\cdots$ and the second by $Z_1Z_2Z_3Z_4\cdots$. Hence W_1 and W_2 require, as shown in [34], only two local settings, while the number of settings needed for $\widetilde{W}$ and W_3 scales exponentially with n (at most we need 3^n setting for $\widetilde{W}$ and 2^n measurement settings for W_3) [33].

It is worth noting that the witness introduced in (9) can be measured by the same local settings needed for the measurements of W_1 and W_2 (and of course by those used for W_3 and $\widetilde{W}$).

Moreover, the advantage given by a low number of measurement settings for the witness W_1 and W_2 is paid for by their weakness with respect to resistance to noise. This will be shown in the following section.

III. RESISTANCE TO NOISE

The strength of a witness is often measured by its resistance to noise, i.e., the amount of noise that can be added to

TABLE I. Traces of the witness operators defined in the text and their corresponding resistance to white noise. The number p_M, depending on the dimension D of the Hilbert space, represents the maximum amount of noise tolerated by the corresponding hyperentanglement witness.

Witness	Tr (W)	p_M
W_1	$(N-1)D$	$\frac{1}{N}=\frac{1}{\log_2 D}\geqslant 0$
W_2 (even N)	$3D-4\sqrt{D}$	$\frac{1}{4}\frac{1}{1-1/\sqrt{D}}\geqslant\frac{1}{4}$
W_2 (odd N)	$3D-3\sqrt{2D}$	$\frac{1}{4}\frac{1}{1-3/4\sqrt{2D}}\geqslant\frac{1}{4}$
W_3 (even N)	$2D-\frac{3D}{(\sqrt{D})^{\log_2 3}}$	$\frac{1}{3}\frac{1}{1-1/(\sqrt{D})^{\log_2 3}}\geqslant\frac{1}{3}$
W_3 (odd N)	$2D-\frac{3D}{2(\sqrt{D/2})^{\log_2 3}}$	$\frac{1}{3}\frac{1}{1-1/2(\sqrt{D/2})^{\log_2 3}}\geqslant\frac{1}{3}$
$\widetilde{W}$	$D-2$	$\frac{1}{2}\frac{1}{1-1/D}\geqslant\frac{1}{2}$

the entangled state in such a way that the witness still measures it as entangled. Consider the states

$$\rho=(1-p_{\text{noise}})|\Xi\rangle\langle\Xi|+p_{\text{noise}}\frac{1}{D}\mathbb{1}, \tag{20}$$

where we have defined $D=2^N$ and p_{noise} measures the amount of (white) noise present in the state. The expectation value of a generic witness (normalized such that $\langle\Xi|W|\Xi\rangle=-1$) is given by

$$\text{Tr}(W\rho)=-1+p_{\text{noise}}+p_{\text{noise}}\frac{1}{D}\text{Tr}(W), \tag{21}$$

so the state is entangled if

$$p_{\text{noise}}<\frac{D}{\text{Tr}(W)+D}\equiv p_M, \tag{22}$$

where p_M is the maximum allowed amount of noise. The trace of W is thus a good measurement of the weakness of the witness: the lower the trace the stronger the resistance to noise. In Table I we show the traces of the above defined witnesses in order of increasing resistance to noise. While $\widetilde{W}$ is highly resistant to white noise (p_M is always greater than 50%) but requires many measurement settings, W_1 and W_2 require only two measurement settings but are less resistant to noise. For any value of N the resistance to noise of our defined witness (19) is larger than that of the witnesses W_2 and W_1, and W_3 can be seen as a compromise between the need to lower the number of settings and that to increase the noise resistance. In general (for any N) the noise tolerance of W_3 is at least 33%. In conclusion, the resistance to noise of the witness grows with the number of settings needed to evaluate it.

IV. NEW ENTANGLEMENT WITNESS FOR GENERIC GRAPH STATES

The same witness defined in (19) can be used for a generic graph state $|G_N\rangle$ associated to a connected graph of N vertices. The graph state is defined by the stabilizer equation as

$$S_k|G_N\rangle=|G_N\rangle, \quad \forall\ k=1,\ldots,N, \tag{23}$$

where

$$S_k=X_k\prod_{j\in\mathcal{N}_k}Z_j, \quad k=1,\ldots,N. \tag{24}$$

Here X_k (Z_k) are the Pauli matrix σ_x (σ_z) acting on qubit k and $\mathcal{N}_k$ is the set of qubits to which it is linked. As shown in [3], for a given connected graph, the following witness detects the genuine N-qubit entanglement of the corresponding graph state[2] $|G_N\rangle$:

$$\widetilde{W}=\mathbb{1}-2|C_N\rangle\langle C_N|. \tag{25}$$

Each state of the form $|\phi_1\rangle_{\mathcal{A}}|\phi_2\rangle_{\mathcal{B}}$ (where $\mathcal{A},\mathcal{B}$ is a generic partition of the N qubits) has a positive expectation value of this witness.

Following the same arguments of the previous section it is possible to show that the following operator is a witness:

$$W_3=\begin{cases}2-3\prod_{j=1}^{n}\left(\frac{1+S_{2j-1}+S_{2j}}{3}\right), & N=2n,\\ 2-3\prod_{j=1}^{n}\left(\frac{1+S_{2j-1}+S_{2j}}{3}\right)\frac{1+S_N}{2}, & N=2n+1.\end{cases} \tag{26}$$

The number of settings needed for $\widetilde{W}$ scales exponentially with the number of qubits [33], while we need more than two measurement settings to determine W_3. The maximum amount of noise p_M allowed for W_3 is shown in Table I.

V. CONCLUSIONS

In this paper we have introduced a method to detect whether a two-particle state is hyperentangled. The method is based on the initial detection of entanglement for each separate degree of freedom. Once this condition is satisfied, the negative value of a *hyperentanglement witness* operator detects the hyperentanglement. We introduced four different hyperentanglement witnesses, namely, $\widetilde{W}$, W_1, W_2, and W_3. They are characterized by different values of the resistance to noise, which grows with the number of measurement settings needed for their evaluation. Precisely, only two local settings are required in the case of W_1 and W_2, while for W_3 and $\widetilde{W}$ their number scales exponentially with the number of DOF's. It is worth noting that the local setting used to measure $\widetilde{W}$, W_1, W_2, and W_3 can also be used to measure the

[2]This witness is slightly different from that defined in [3], due to the normalization constant needed to satisfy (7).

witnesses $\mathcal{W}^{(j)}$ [see Eq. (9)] and then reveal the entanglement corresponding to each DOF separately. For example, with the local setting $Z_1Z_2Z_3Z_4Z_5Z_6\cdots$ it is possible to measure the observables $S_2, S_4, S_6, \ldots$ and all their products.

A low number of measurements corresponds to a reduced resistance to noise. Indeed, it is shown that the amount of white noise tolerated by W_3 and $\widetilde{W}$ exceeds those tolerated by W_1 and W_2 (see Table I).

In general, a hyperentangled state (2) can also be expressed as a maximally entangled state of two qudits, where $d=2^n$, since each particle encodes n qubits. However, our approach is different from the usual bipartite qudit entanglement since the witness detecting the bipartite entanglement between the two particles is (up to normalization) $W=\mathbb{1}/2^n - |\Xi\rangle\langle\Xi|$, because the maximum overlap between a maximally entangled state of two qudits and a separable state is exactly $1/2^n$. The witness (15) is far more stringent since many entangled states in the bipartite sense are not hyperentangled. For example the state $|\psi_1\rangle$ in (12) can be written as $|00\rangle_A|00\rangle_B+|01\rangle_A|01\rangle_B$. This state clearly shows entanglement between A and B but it is not hyperentangled.

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APPENDIX: DEMONSTRATION

Let the state $|\varphi\rangle$ satisfy Eq. (8). This means that j must exist such that $|\varphi\rangle$ can be written as

$$\begin{aligned}|\varphi\rangle &= |\varphi_1\rangle_{A_j\mathcal{I}}|\varphi_2\rangle_{B_j\mathcal{J}}\\ &= (a|0\rangle_{A_j}|\chi_1\rangle_{\mathcal{I}} + b|1\rangle_{A_j}|\chi_2\rangle_{\mathcal{I}}) \otimes (c|0\rangle_{B_j}|\chi_3\rangle_{\mathcal{J}} + d|1\rangle_{B_j}|\chi_4\rangle_{\mathcal{J}})\end{aligned} \tag{A1}$$

with $|\chi_i\rangle$ normalized and $|a|^2+|b|^2=|c|^2+|d|^2=1$. Let us define as $\gamma_{\mu\nu}$ the overlap between $|\chi_\mu\rangle_{\mathcal{I}}|\chi_\nu\rangle_{\mathcal{J}}$ and $\langle\phi^+|_{A_1B_1}\cdots\langle\phi^+|_{A_{j-1}B_{j-1}}\langle\phi^+|_{A_{j+1}B_{j+1}}\cdots\langle\phi^+|_{A_nB_n}$. Since the states $|\chi_i\rangle$ are normalized we have $|\gamma_{\mu\nu}|^2\leqslant 1$. By calculating the overlap we obtain

$$\begin{aligned}|\langle\Xi|\varphi\rangle|^2 &= |\langle\phi^+|(ac\gamma_{13}|00\rangle + ad\gamma_{14}|01\rangle\\ &\quad + bc\gamma_{23}|10\rangle + bd\gamma_{24}|11\rangle)_{A_jB_j}|^2\\ &= \frac{1}{2}|(ac\gamma_{13} + bd\gamma_{24})|^2 \leqslant \frac{1}{2}(|ac\gamma_{13}|^2 + |bd\gamma_{24}|^2)\\ &\leqslant \frac{1}{2}(|ac|^2 + |bd|^2) \leqslant \frac{1}{2}(|a|^2 + |b|^2) = \frac{1}{2},\end{aligned} \tag{A2}$$

where we used $|\gamma_{\mu\nu}|^2$, $|c|^2$, $|d|^2\leqslant 1$ and the property $|r+s|^2 \leqslant |r|^2+|s|^2$. The bound is easily saturated, for example, by the states of the form

$$|\phi^+\rangle_{A_1B_1}\cdots|\phi^+\rangle_{A_{j-1}B_{j-1}}|0\rangle_{A_j}|0\rangle_{B_j}|\phi^+\rangle_{A_{j+1}B_{j+1}}\cdots|\phi^+\rangle_{A_nB_n}. \tag{A3}$$

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