An Optical Scheme for the Generation and Analysis of a Two Photon Six-Qubit Linear Cluster State

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We present the project of an optical scheme that allows to engineer a 2-photon 6-qubit linear cluster state. By this scheme we are able to create a hyperentangled state based on polarization, linear momentum and arrival time entanglement. Since the three degrees of freedom of the photons are fully independent, the hyperentangled state allows the maximum entanglement of the particles. By suitable local unitary operations the hyperentangled state can be easily transformed into a linear cluster state, with important applications in photonic quantum computation and quantum nonlocality tests.

1. REALIZATION OF THE $|LC_6\rangle$ STATE

Measurement-based quantum computation requires the generation of a particular class of $n$-qubit entangled states, the cluster states.1,2 The number $n$ of qubits requested in a cluster state becomes larger and larger with the increase of complexity of quantum algorithms. The implementation of cluster states with $n$ qubits encoded in some degrees of freedom of $n$ photons can be extremely difficult due to the low efficiency of multi-photon interactions. Indeed, the experimental preparations of 6-qubit 6-photon cluster states realized so far3 presented low entanglement visibilities. We present here the proposal of an experimental setup allowing to generate and analyze a 6-qubit entanglement visibilities. We present here the proposal of an optical scheme that allows the maximum entanglement of the particles. By suitable local unitary operations the hyperentangled state can be easily transformed into a linear cluster state, with important applications in photonic quantum computation and quantum nonlocality tests.

The proposed experimental setup is an evolution of the two-photon six-qubit linear cluster state. The LC equivalence between $|LC_6\rangle$ and $|\psi\rangle$ is defined local unitary (LU) equivalent if there exists an LU operation $U_L$ which maps $|\psi\rangle$ to $|\psi\rangle$. Two $n$-qubit states $|\psi\rangle$ and $|\psi'\rangle$ are said to be local Clifford (LC) equivalent if they are LU equivalent, and the $U_L$ operation, which transforms $|\psi'\rangle$ into $|\psi\rangle$, maps the Pauli group into itself.

$|LC_6\rangle = U_L |\psi\rangle$

In the previous equations, $CX_{65}$ represents the Controlled-$X$ gate between qubits 6 (control) and 5 (target), while $H_j$ the Hadamard gate $H_j = (1/\sqrt{2}) (X + Z)$ acting on qubit $j$. The LC equivalence between $|LC_6\rangle$ and $|\psi\rangle$ allows a one-to-one correspondence between the measurement observables on $|LC_6\rangle$ and on $|\psi\rangle$ states (see Table I).

The proposed experimental setup is an evolution of the two-photon hyper-entangled (in momentum and polarization) state
source shown in Figure 2. The state generated by this source is expressed in the following way (see Refs. [4–6] for more details):

\[
\frac{1}{\sqrt{2}}(|H_A H_B + V_A V_B|) \otimes \frac{1}{\sqrt{2}}(|r_A l_B + l_A r_B|)
\]

where \(|H_j\) (\(|V_j\)) and \(|r_j\) (\(|l_j\)) respectively represent the horizontal (vertical) polarization and the left (right) path of the photon \(j\). In order to produce \(|C_6\) it is necessary to encode two more qubits in an additional degree of freedom. For this purpose, we used nonlinear crystals, which are simply rejected by the coincidence measurement. A scheme similar to the one presented in this article has been recently proposed for a conclusive energy-time Bell test of local realism.\(^7\)

Going back to the scheme conceived for the generation of the 6-qubit hyperentangled state, the basic idea is to inject the state created by the source of Figure 2(left) into the Michelson interferometer. After the first passage of photons through the BS the following state is generated:

\[
\frac{1}{\sqrt{2}}(|L_A L_B + S_A S_B|) \otimes \frac{1}{\sqrt{2}}(|H_A H_B + V_A V_B|)
\]

Under the correspondence \(|L_A L_B \leftrightarrow |01\rangle, |S_A S_B \leftrightarrow |11\rangle, |H_A H_B \leftrightarrow |02\rangle, |V_A V_B \leftrightarrow |12\rangle\), we note that by performing a \(C_{X65}\) and a \(CZ_{12}\) to our state, the generated state is equivalent to \(|C_6\) expressed in the logical basis \(0\) and \(1\).\(^8\) It is worth stressing that the two controlled operations act only on single photon degrees of freedom. In this sense they are local and can be easily implemented by using single-photon detectors. At the same time these operations act on the two photons. They are emitted at degenerate wavelength by spontaneous parametric down conversion SPDC (pumped by a continuous wave laser) in a Type I phase matched nonlinear (NLO) crystal and injected into the MI.

In this setup the same beam splitter (BS) is used both for the preparation of the state in the first passage of the photons and for the final measurement in the second passage. Photons, whose directions are carefully selected by two masks, can travel along the Short (\(S\)) or Long (\(L\)) path, after the first partition occurring on the BS. After reflection by mirrors, each photon experiences a new BS reflection-transmission process with relative path delay \(\Delta t_0 = 2L - S\). The BS output beams are then detected by single photon detectors. The relative phase of the path contributions \(|S_A, S_B\) and \(|L_A, L_B\) can be tuned by a piezo transducer that finely changes the mirror position in the long arm. Interesting enough, this single MI configuration allows to minimize the state phase instabilities.

The key element of this setup is represented by the swapping operation between the two photon paths, performed in the \(L\) arm of MI. It is realized by a lens (see Fig. 2(right)) aligned at a distance exactly equal to its focal length from mirror. By this configuration we are able to detect coincidences only if both photons travel along the same arm of MI after the first interaction with BS. In fact if, for instance, the first photon passes through the \(L\) path and the second follows the \(S\) path (i.e., if we consider the event \(|L_A S_B\rangle\), after the second interaction with BS they will end on the same detector. Thus no coincidence is detected in this case. The same happens with the \(|S_A L_B\rangle\) event.

By this setup it is no longer necessary to perform the nonlocal temporal post-selection which is needed in the Franson’s scheme to discard the events \(|L_A S_B\rangle\) and \(|S_A L_B\rangle\). By our scheme they are simply rejected by the coincidence measurement. A scheme similar to the one presented in this article has been recently proposed for a conclusive energy-time Bell test of local realism.\(^7\)

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### Table I. LC equivalence between \(|C_6\) and \(|LC_6\) Pauli operators.

| photons  | \(|C_6\) | \(|LC_6\) | \(|C_6\) | \(|C_6\) |
|----------|---------|---------|---------|---------|
| \(L_{C6}\) | \(X_1\) | \(Y_1\) | \(Z_1\) | \(Z_2\) |
| \(L_{C6}\) | \(Y_1\) | \(Y_1\) | \(Z_1\) | \(Z_2\) |
| \(L_{C6}\) | \(Z_1\) | \(Z_1\) | \(Z_1\) | \(Z_2\) |
| \(L_{C6}\) | \(Z_2\) | \(Z_2\) | \(Z_2\) | \(Z_2\) |
| \(L_{C6}\) | \(Y_2\) | \(Y_2\) | \(Y_2\) | \(Y_2\) |
| \(L_{C6}\) | \(Z_2\) | \(Z_2\) | \(Z_2\) | \(Z_2\) |
| \(L_{C6}\) | \(X_3\) | \(X_3\) | \(X_3\) | \(X_3\) |
| \(L_{C6}\) | \(Y_3\) | \(Y_3\) | \(Y_3\) | \(Y_3\) |
| \(L_{C6}\) | \(Z_3\) | \(Z_3\) | \(Z_3\) | \(Z_3\) |

\*As shown in the appendix, the cluster state can be equivalently generated by applying a \(CZ_{60}\) and a \(CX_{12}\) transformation.

### 2. ANALYSIS OF THE \(|LC_6\) STATE

The measurement apparatus proposed to analyze the \(|C_6\) state is sketched in Figure 3. In the scheme we have included the MI interferometer that creates the time-energy entanglement. As a matter of fact its generation takes place just before the second passage of the photons through the BS, while the interferometer output state corresponds to the projection of \(|C_6\) on the \(|L_A L_B \pm |S_A S_B\rangle\) basis. Two thin glass plates suitably inserted on the MI short arm allow the projection on the \(|L_A L_B \pm |S_A S_B\rangle\) basis. The
measurement in the standard computational basis can be easily implemented by blocking the photons that pass through one of the two MI arms.

The momentum measurements are performed in a very similar way. The two mode pairs \( \ell_A - r_B \) and \( \ell_A - r_B \) emerging from the MI, are spatially and temporally combined onto a 50% beam splitter (BS') by an interferometric apparatus, where a trombone mirror assembly allows fine delay adjustment \( \Delta x \). By suitable tilting of two thin glass plates inserted on one arm of the interferometer, we can separately set the relative phase of the A and B modes matching on BS'.

Polarization analyses are performed by standard tomographic apparatus on each detector arm, given by a sequence of \( \lambda/4 \) waveplates and a polarizing beam splitter (PBS).

3. CONCLUSIONS

In this article we proposed an optical scheme for the generation of a 6-qubit 2-photon cluster state. The method is based on the local manipulation of a polarization-momentum-energy/time hyperentangled state generated by a SPDC source. The cluster states can be generated at high repetition rate thanks to the high production rate (\( \sim 10 \text{ KHz} \)) of the photon pair. The \(|LC_6\rangle\) cluster state allows to span a high dimension Hilbert space. Because of this reason it has relevant applications both for the one-way model of quantum computation and for advanced tests of quantum mechanics since it has been demonstrated that the quantum nonlocality of a state grows with its size.

4. APPENDIX: CALCULATIONS

In this section we report some useful calculations of equivalent expressions of the \(|LC_6\rangle\) state. Remembering that:

\[
\begin{align*}
\text{CZ}_1|+\rangle,|+\rangle & = \frac{1}{\sqrt{2}}(|0\rangle,|+\rangle) + |1\rangle,|+\rangle) \\
& = \frac{1}{\sqrt{2}}(|+\rangle,|0\rangle) + |1\rangle,|1\rangle) = \text{CZ}_1|+\rangle,|+\rangle \\
\end{align*}
\]
we have:

\[
|LC_6\rangle = CZ_{14}CZ_{12}CZ_{25}CZ_{56}CZ_{36}|+_1+_2+_3+_4+_5+_6
\]

\[
= CZ_{12}CZ_{56} \frac{1}{2\sqrt{2}}((0)_1|+_4 + |1)_1|_-4)
\]

\[
\otimes ((0)_2|+_3 + |1)_2|_-3) \otimes ((0)_3|+_6 + |1)_3|_-6)
\]

(7)

If we define \( |\phi^+\rangle_{ij} \equiv 1/\sqrt{2}(|0\rangle_i|0\rangle_j + |1\rangle_i|1\rangle_j) \), from the last expression it is easy to check the equivalence of the following ones:

\[
|LC_6\rangle = CZ_{12}CZ_{56}H_1H_2H_3|\phi^+\rangle_{14} \otimes |\phi^+\rangle_{25} \otimes |\phi^+\rangle_{36}
\]

(8)

By using the relations

\[
ZH = HX,
\]

\[
CZ_{ij} = \frac{1}{2} + \frac{Z_i}{2} \otimes 1_j + \frac{1}{2} - \frac{Z_j}{2} \otimes Z_i
\]

(9)

we can show that

\[
CZ_{ij}H_j = H_jCZ_{ij},
\]

\[
CZ_{ij}H_i = H_iCZ_{ij}
\]

(10)

where \( CZ_{ij} \) stands for the controlled-Z transformation acting on qubit \( i \) when the control qubit \( j \) is in the state \( |0\rangle \) instead of \( |1\rangle \).

The previous equations can be used to show that

\[
|LC_6\rangle = (H_1H_2H_3)CZ_{12}CZ_{56} \frac{1}{2\sqrt{2}}((0)_1|0\rangle_4 + |1\rangle_1|1\rangle_4)
\]

\[
\otimes ((0)_2|0\rangle_3 + |1\rangle_2|1\rangle_3) \otimes ((0)_3|0\rangle_6 + |1\rangle_3|0\rangle_6)
\]

(11)

\[
|LC_6\rangle = (H_1H_2H_3)CZ_{12}CZ_{56} \frac{1}{2\sqrt{2}}((0)_1|0\rangle_4 + |1\rangle_1|1\rangle_4)
\]

\[
\otimes ((0)_2|0\rangle_3 + |1\rangle_2|1\rangle_3) \otimes ((0)_3|0\rangle_6 + |1\rangle_3|0\rangle_6)
\]

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References and Notes


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